Ribbon graphs (B. Bollobás and O. Riordan [BR2, BR3])

A ribbon graph $G$ is a surface represented as union of vertices-discs and edges-ribbons such that

- discs and ribbons intersect by disjoint line segments,
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.

The Bollobás-Riordan polynomial [BR2, BR3]

- $v(G)$ be the number of vertices of $G$;
- $e(G)$ be the number of its edges of $G$;
- $k(G)$ be the number of components of $G$;
- $r(G) := v(G) - k(G)$ be the rank of $G$;
- $n(G) := e(G) - r(G)$ be the nullity of $G$;
- $bc(G)$ be the number of connected components of the boundary of $G$;

$$R_G(x, y, z) := \sum_F x^{r(G) - r(F)} y^{n(F)} z^{k(F) - bc(F) + n(F)}$$

Example

\[ \begin{array}{cccc}
(k, r, n, bc) & (2, 0, 0, 2) & (1, 1, 0, 1) & (1, 1, 0, 1) & (2, 0, 1, 2) \\
\hline
& (1, 1, 1, 2) & (1, 1, 1, 1) & (1, 1, 1, 1) & (1, 1, 2, 1) \\
\end{array} \]

$$R_G(x, y, z) = x + 2 + xyz + y + 2yz + y^2z^2.$$  

Relations to the Tutte polynomial

$R_G(x - 1, y - 1, 1) = T_G(x, y)$. If $G$ is planar (genus zero): $R_G(x - 1, y - 1, z) = T_G(x, y)$.

References