## Jones polynomial

### The Kauffman bracket and the Jones polynomial [Ka1].

Let $L$ be a link diagram.

- **A-splitting:**
  
  ![A-splitting diagram]

- **B-splitting:**
  
  ![B-splitting diagram]

A state $S$ is a choice of either $A$- or $B$-splitting at every classical crossing.

- $\alpha(S) = \#(\text{of } A\text{-splittings in } S)$
- $\beta(S) = \#(\text{of } B\text{-splittings in } S)$
- $\delta(S) = \#(\text{of circles in } S)$

$$\mathcal{L}(A, B, d) := \sum_S A^{\alpha(S)} B^{\beta(S)} d^{\delta(S)-1}$$

$$J_L(t) := (-1)^{w(\mathcal{L})} t^{3w(\mathcal{L})/4} \mathcal{L}(t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2})$$

### Example

<table>
<thead>
<tr>
<th>((\alpha, \beta, \delta))</th>
<th>((3, 0, 1))</th>
<th>((2, 1, 2))</th>
<th>((2, 1, 2))</th>
<th>((1, 2, 1))</th>
</tr>
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<tbody>
<tr>
<td>((2, 1, 2))</td>
<td>((1, 2, 1))</td>
<td>((1, 2, 3))</td>
<td></td>
<td>((0, 3, 2))</td>
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$$\mathcal{L} = A^3 + 3A^2Bd + 2AB^2 + AB^2d^2 + B^3d; \quad J_L(t) = 1$$

### Thistlethwaite’s Theorem [Ka1]

Up to a sign and multiplication by a power of $t$ the Jones polynomial $J_L(t)$ of an alternating link $L$ is equal to the Tutte polynomial $T_T(-t, -t^{-1})$.

- ![Thistlethwaite’s Theorem Diagram](image.png)

The theorem was generalized to non-alternating links using signed graphs in [Ka2] and using the Bollobás-Riordan polynomial for ribbon graphs in [DFKLS]; and to virtual links in [ChVo, Ch].

### Theorem [Ch]

Let $L$ be a virtual link diagram with $e$ classical crossings, $G_L^s$ be the signed ribbon graph corresponding to a state $s$, and $v := v(G_L^s)$, $k := k(G_L^s)$. Then $e = e(G_L^s)$ and

$$\mathcal{L}(A, B, d) = A^e \left( x^k y^v z^{x+1} R_{G_L^s}(x, y, z) \right) \bigg|_{x = \frac{a_d}{\delta}, \ y = \frac{b_d}{\delta}, \ z = \frac{1}{\delta}}.$$
Diagram

State $s$

Attaching planar bands

Replacing bands by arrows

Untwisting state circles

Pulling state circles apart

Forming the ribbon graph $G_L^s$

REFERENCES


