Graphs

**Definition.** A graph $G$ is a finite set of vertices $V(G)$ and a finite set $E(G)$ of unordered pairs $(x, y)$ of vertices $x, y \in V(G)$ called edges.

A graph may have loops $(x, x)$ and multiple edges when a pair $(x, y)$ appears in $E(G)$ several times. Pictorially we represent the vertices by points and edges by lines connecting the corresponding points. Topologically a graph is a 1-dimensional cell complex with $V(G)$ as the set of 0-cells and $E(G)$ as the set of 1-cells. Here are two pictures representing the same graph.

![Graphs](image)

**Tutte polynomial**

**Chromatic polynomial** $C_G(q)$.

A coloring of $G$ with $q$ colors is a map $c : V(G) \to \{1, \ldots, q\}$. A coloring $c$ is proper if for any edge $e$: $c(v_1) \neq c(v_2)$, where $v_1$ and $v_2$ are the endpoints of $e$.

**Definition 1.** $C_G(q) := \# \text{ of proper colorings of } G \text{ in } q \text{ colors}$.

**Properties (Definition 2).**

$C_G = C_{G-e} - C_{G/e}$;  
$C_{G_1 \sqcup G_2} = C_{G_1} \cdot C_{G_2}$, for a disjoint union $G_1 \sqcup G_2$;  
$C_\ast = q$.

**Tutte polynomial** $T_G(x, y)$.

**Definition 1.**

$T_G = T_{G-e} + T_{G/e}$ if $e$ is neither a bridge nor a loop;  
$T_G = xT_{G/e}$ if $e$ is a bridge;  
$T_G = yT_{G-e}$ if $e$ is a loop;  
$T_{G_1 \sqcup G_2} = T_{G_1} \cdot T_{G_2}$ for a disjoint union $G_1 \sqcup G_2$ and a one-point join $G_1 \cdot G_2$;  
$T_\ast = 1$.

**Properties.**

$T_G(1, 1)$ is the number of spanning trees of $G$;  
$T_G(2, 1)$ is the number of spanning forests of $G$;  
$T_G(1, 2)$ is the number of spanning connected subgraphs of $G$;  
$T_G(2, 2) = 2^{|E(G)|}$ is the number of spanning subgraphs of $G$.  
$C_G(q) = q^{k(G)}(-1)^{r(G)}T_G(1 - q, 0)$.

**Definition 2.**

Let $F$ be a graph:
- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of components of $F$;
• \( r(F) := v(F) - k(F) \) be the rank of \( F \);
• \( n(F) := e(F) - r(F) \) be the nullity of \( F \);

\[
T_G(x, y) := \sum_{F \subseteq E(G)} (x - 1)^{r(G) - r(F)} (y - 1)^{n(F)}
\]

Dichromatic polynomial \( Z_G(q, v) \) (Definition 3).

Let \( \text{Col}(G) \) denote the set of colorings of \( G \) with \( q \) colors, and let \( D : E(G) \times \text{Col}(G) \to \{0, 1\} \) be defined by the formula \( D(e, c) = 1 \) is and only if \( c(v_1) = c(v_2) \), where \( v_1 \) and \( v_2 \) are the endpoints of \( e \).

\[
Z_G(q, v) := \sum_{c \in \text{Col}(G)} \prod_{e \in E(G)} \left( 1 + v D(e, c) \right)
\]

Properties .
\[
Z_G = Z_{G-e} + v Z_{G/e};
Z_{G_1 \sqcup G_2} = Z_{G_1} \cdot Z_{G_2}; \quad \text{for a disjoint union } G_1 \sqcup G_2;
Z_\ast = q;
Z_G(q, v) = \sum_{F \subseteq E(G)} q^{k(F)} v^{e(F)};
C_G(q) = Z_G(q, -1);
Z_G(q, v) = q^{k(G)} v^{e(G)} T_G(1 + qv^{-1}, 1 + v);
T_G(x, y) = (x - 1)^{-k(G)} (y - 1)^{-v(G)} Z_G((x - 1)(y - 1), y - 1).
\]

Potts model in statistical mechanics (Definition 4).

Potts model (C.Domb 1952): \( q = 2 \) the Ising model (W.Lenz, 1920)

Let \( G \) be a graph.

Particles are located at vertices of \( G \). Each particle has a spin, which takes \( q \) different values. A state, \( \sigma \in \mathfrak{S} \), is an assignment of spins to all vertices of \( G \). Neighboring particles interact with each other only if their spins are the same.

The energy of the interaction along an edge \( e \) is \( -J_e \) (coupling constant). The model is called ferromagnetic if \( J_e > 0 \) and antiferromagnetic if \( J_e < 0 \).

Energy of a state \( \sigma \) (Hamiltonian),
\[
H(\sigma) = - \sum_{(a, b) = e \in E(G)} J_e \delta(\sigma(a), \sigma(b)).
\]

Boltzmann weight of \( \sigma \):
\[
e^{-\beta H(\sigma)} = \prod_{(a, b) = e \in E(G)} e^{J_e \beta \delta(\sigma(a), \sigma(b))} = \prod_{(a, b) = e \in E(G)} \left( 1 + (e^{J_e \beta} - 1)\delta(\sigma(a), \sigma(b)) \right),
\]
where the inverse temperature \( \beta = \frac{1}{\kappa T} \), \( T \) is the temperature, \( \kappa = 1.38 \times 10^{-23} \) joules/Kelvin is the Boltzmann constant.
The Potts partition function (for $x_e := e^{J_e \beta} - 1$)

$$Z_G(q, x_e) := \sum_{\sigma \in \mathcal{S}} e^{-\beta H(\sigma)} = \sum_{\sigma \in \mathcal{S}} \prod_{e \in E(G)} \left( 1 + x_e \delta(\sigma(a), \sigma(b)) \right)$$

Properties of the Potts model

Probability of a state $\sigma$: $P(\sigma) := e^{-\beta H(\sigma)} / Z_G$.

Expected value of a function $f(\sigma)$:

$$\langle f \rangle := \sum_{\sigma} f(\sigma) P(\sigma) = \sum_{\sigma} f(\sigma) e^{-\beta H(\sigma)} / Z_G$$

Expected energy: $\langle H \rangle = \sum_{\sigma} H(\sigma) e^{-\beta H(\sigma)} / Z_G = -\frac{d}{d\beta} \ln Z_G$.

Fortuin—Kasteleyn’1972: $Z_G(q, x_e) = \sum_{F \subseteq E(G)} q^{k(F)} \prod_{e \in F} x_e$, where $k(F)$ is the number of connected components of the spanning subgraph $F$.

$Z_G = Z_G \setminus e + x_e Z_G / e$.

Spanning tree generating function (Definition 5).

For a connected graph $G$ fix an order of its edges: $e_1, e_2, \ldots, e_m$. Let $T$ be a spanning tree.

An edge $e_i \in E(T)$ is called internally active (live) if $i < j$ for any edge $e_j$ connecting the two components of $T - e_i$.

An edge $e_j \not\in E(T)$ is called externally active (live) if $j < i$ for any edge $e_i$ in the unique cycle of $T \cup e_j$.

Let $i(T)$ and $j(T)$ be the numbers of internally and externally active edges correspondingly.

$$T_G(x, y) := \sum_{T} x^{i(T)} y^{j(T)}$$