HOMFLYPT polynomial

The HOMFLY polynomial $P(L)$ is defined as a Laurent polynomial in two variables $a$ and $z$ with integer coefficients satisfying the following skein relation and the initial condition:

$$aP(i\begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\draw[thick] (0,0) -- (1,1);
\draw[thick] (0,1) -- (1,0);
\end{tikzpicture}) - a^{-1}P(i\begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\draw[thick] (0,0) -- (1,1);
\draw[thick] (0,1) -- (1,0);
\end{tikzpicture}) = zP(i\begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\end{tikzpicture}) ; \quad P(\begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\end{tikzpicture}) = 1 .$$

The existence of such an invariant is a difficult theorem. It was established simultaneously and independently by five groups of authors [HOM, PT] (see also [Lik]). The HOMFLY polynomial is equivalent to the collection of quantum invariants associated with the Lie algebra $\mathfrak{sl}_N$ and its standard $N$-dimensional representation for all values of $N$.

Examples.

$$P\left(\begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\draw[thick] (0,0) -- (1,1);
\draw[thick] (0,1) -- (1,0);
\end{tikzpicture}\right) = (2a^2 - a^4) + a^2z^2 , \quad P\left(\begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\draw[thick] (0,0) -- (1,1);
\draw[thick] (0,1) -- (1,0);
\end{tikzpicture}\right) = (a^{-2} - 1 + a^2) - z^2 .$$

Properties.

(1) HOMFLYPT polynomial of a knot is preserved when the knot orientation is reversed.
(2) $P(\bar{L}) = P(L)$, where $\bar{L}$ is the mirror reflection of $L$ and $\bar{P}(L)$ is the polynomial obtained from $P(L)$ by substituting $a^{-1}$ for $a$;
(3) $P(K_1 \# K_2) = P(K_1) \cdot P(K_2)$;
(4) $P(L_1 \sqcup L_2) = \frac{a-a^{-1}}{2} \cdot P(L_1) \cdot P(L_2)$;
(5) $P(8_8) = P(10_{129})$ and $P(C) = P(KT)$ for the Conway, $C$, and the Kinoshita–Terasaka, $KT$, knots below.

$$8_8 = \begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\draw[thick] (0,0) -- (1,1);
\draw[thick] (0,1) -- (1,0);
\end{tikzpicture} , \quad 10_{129} = \begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\draw[thick] (0,0) -- (1,1);
\draw[thick] (0,1) -- (1,0);
\end{tikzpicture} , \quad C = \begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\draw[thick] (0,0) -- (1,1);
\draw[thick] (0,1) -- (1,0);
\end{tikzpicture} , \quad KT = \begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\draw[thick] (0,0) -- (1,1);
\draw[thick] (0,1) -- (1,0);
\end{tikzpicture} .$$

Two-variable Kauffman polynomial

L. Kauffman [Ka] found another invariant Laurent polynomial $F(L)$ in two variables $a$ and $z$. Firstly, for a unoriented link diagram $D$ we define a polynomial $\Lambda(D)$ which is invariant under Reidemeister moves II and III and satisfies the skein relations

$$\Lambda(i\begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\draw[thick] (0,0) -- (1,1);
\draw[thick] (0,1) -- (1,0);
\end{tikzpicture}) + \Lambda(i\begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\draw[thick] (0,0) -- (1,1);
\draw[thick] (0,1) -- (1,0);
\end{tikzpicture}) = z\left(\Lambda(i\begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\end{tikpicture}) + \Lambda(i\begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\end{tikpicture})\right) ,$$

$$\Lambda(i\begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\end{tikzpicture}) = a\Lambda(i\begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\end{tikpicture}) , \quad \Lambda(i\begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\end{tikzpicture}) = a^{-1}\Lambda(i\begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\end{tikpicture}) ,$$

and the initial condition $\Lambda(\begin{tikzpicture}[baseline=-.5ex]
\fill[fill=white] (0,0) rectangle (1,1);
\end{tikzpicture}) = 1$.

Now, for any diagram $D$ of an oriented link $L$ we put

$$F(L) := a^{-w(D)}\Lambda(D) .$$

The Kauffman polynomial is equivalent to the collection of the quantum invariants associated with the Lie algebra $\mathfrak{so}_N$ and its standard $N$-dimensional representation for all values of $N$. 
Examples.

\[ F\left( \begin{array}{c}
\end{array} \right) = (-2a^2 - a^4) + (a^3 + a^5)z + (a^2 + a^4)z^2, \]

\[ F\left( \begin{array}{c}
\end{array} \right) = (-a^{-2} - 1 - a^2) + (-a^{-1} - a)z + (a^{-2} + 2 + a^2)z^2 + (a^{-1} + a)z^3. \]

Properties.

1. \( F(K) \) is preserved when the knot orientation is reversed.
2. \( F(L) = F(L) \), where \( L \) is the mirror reflection of \( L \), and \( F(L) \) is the polynomial obtained from \( F(L) \) by substituting \( a^{-1} \) for \( a \);
3. \( F(K_1 \# K_2) = F(K_1) \cdot F(K_2) \);
4. \( F(L_1 \sqcup L_2) = \left( (a + a^{-1})z^{-1} - 1 \right) \cdot F(L_1) \cdot F(L_2) \);
5. (these knots can be distinguished by the Conway and, hence, by the HOMFLY polynomial).

Vassiliev knot invariants

The main idea of the combinatorial approach to the theory of Vassiliev knot invariants, also known as finite type invariants, is to extend a knot invariant \( v \) to singular knots with double points according to the following rule, which we will refer to as Vassiliev skein relation:

\[ v\left( \begin{array}{c}
\end{array} \right) := v\left( \begin{array}{c}
\end{array} \right) - v\left( \begin{array}{c}
\end{array} \right). \]

Definition. A knot invariant is said to be a Vassiliev invariant of order (or degree) \( \leq n \) if its extension vanishes on all singular knots with more than \( n \) double points.

Denote by \( V_n \) the set of Vassiliev invariants of order \( \leq n \) with values in the field of complex numbers \( \mathbb{C} \). The definition implies that, for each \( n \), the set \( V_n \) forms a complex vector space. Moreover, \( V_n \subseteq V_{n+1} \), so we have an increasing filtration

\[ V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n \subseteq \cdots \subseteq V := \bigcup_{n=0}^{\infty} V_n. \]

It will be shown that the spaces \( V_n \) have finite dimension, and that the quotients \( V_n/V_{n-1} \) admit a nice combinatorial description. The study of these spaces is the main purpose of the combinatorial Vassiliev invariant theory. The exact dimension of \( V_n \) is known only for \( n \leq 12 \):

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<tr>
<td>( \dim V_n/V_{n-1} )</td>
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References