Let \( E = \{v_1, \ldots, v_n\} \) be a collection of vectors in a vector space \( U \) and \( M \) be a matroid of their linear dependences. Consider an \( n \)-dimensional vector space \( V \) with a basis \( e_1, \ldots, e_n \) and a linear map \( f : V \to U \) sending \( e_k \) to \( v_k \). Denote the kernel of this map by \( W \). It is a subspace of \( V \) and there is a natural inclusion map \( i : W \hookrightarrow V \). There is the dual map \( W^* \to V^* \) of dual vector spaces. The space \( V^* \) has a natural dual basis \( e_1^*, \ldots, e_n^* \). Their images \( i^*(e_1^*), \ldots, i^*(e_n^*) \) is a collection of vectors in the space \( W^* \). These vectors with the structure of linear dependences between them form the dual matroid \( M^* \).

**Dual representable matroids**

The Las Vergnas polynomial

Reference: M. Las Vergnas [LV].

**Matroid perspectives.**

A bijection \( M \to M' \) is called matroid perspective if any circuit of \( M \) is mapped to a union of circuits of \( M' \). Equivalently,

\[
  r_M(X) - r_M(Y) \geq r_{M'}(X) - r_{M'}(Y) \quad \text{for all } Y \subseteq X.
\]

**Example.**

For graphs \( G \) and \( G^* \) dually embedded in a surface, then the map of the bond matroid of \( G^* \) onto the circuit matroid of \( G, B(G^*) \to C(G) \), is a matroid perspective.

**Definition.**

\[
  T_{M \to M'}(x, y) := \sum_{X \subseteq M} (x - 1)^{r(M') - r(M)(X)}(y - 1)^{n_M(X)}z^{(r(M) - r(M)(X)) - (r(M') - r(M')(X))}
\]

**Properties.**

\[
  T_M(x, y) = T_{M \to M}(x, y, z) ; \\
  T_M(x, y) = T_{M \to M'}(x, y, x - 1) ; \\
  T_{M'}(x, y) = (y - 1)^{r(M) - r(M')}T_{M \to M'}(x, y, \frac{1}{y - 1}) ;
\]

**Ribbons graphs (graphs on surfaces)**

**Definition.** A ribbon graph \( G \) is a surface (possibly non-orientable) with boundary, represented as the union of two sets of closed topological discs called vertices \( V(G) \) and edges \( E(G) \), satisfying the following conditions:

- these vertices and edges intersect by disjoint line segments;
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.
The Bollobás-Riordan polynomial
Reference: B. Bollobás and O. Riordan [BR].

\[ R_G(\{x_e, y_e\}, X, Y, Z) := \sum_{F \subseteq G} \left( \prod_{e \in F} x_e \right) \left( \prod_{e \in F} y_e \right) X^{r(G) - r(F)} Y^{n(F)} Z^{k(F) - bc(F) + n(F)} \]

For signed graphs, we set \( x_+ = 1, \quad x_- = (X/Y)^{1/2}, \)
\( y_+ = 1, \quad y_- = (Y/X)^{1/2}. \)

Example.

<table>
<thead>
<tr>
<th>((k, r, n, bc)) term of (R_G)</th>
<th>((1,1,1,2))</th>
<th>((1,1,0,1))</th>
<th>((1,1,0,1))</th>
<th>((2,0,0,2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X)</td>
<td>((-\quad-\quad-))</td>
<td>((-\quad-))</td>
<td>((-\quad-))</td>
<td>((-\quad-))</td>
</tr>
<tr>
<td>((1,1,2,1)) (XYZ^2)</td>
<td>((-\quad-\quad-))</td>
<td>((-\quad-))</td>
<td>((-\quad-))</td>
<td>((-\quad-))</td>
</tr>
<tr>
<td>((1,1,1,1)) (YZ)</td>
<td>((-\quad-))</td>
<td>((-\quad-))</td>
<td>((-\quad-))</td>
<td>((-\quad-))</td>
</tr>
<tr>
<td>((1,1,1,1)) (YZ)</td>
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</tr>
<tr>
<td>((2,0,1,2)) (Y^2Z)</td>
<td>((-\quad-))</td>
<td>((-\quad-))</td>
<td>((-\quad-))</td>
<td>((-\quad-))</td>
</tr>
</tbody>
</table>

\[ R_G(X, Y, Z) = X + 2 + Y + XYZ^2 + 2YZ + Y^2Z \]

Properties.

\[ R_G = x_e R_{G/e} + y_e R_{G-e} \quad \text{if } e \text{ is ordinary, that is neither a bridge nor a loop}, \]
\[ R_G = (x_e + X y_e) R_{G/e} \quad \text{if } e \text{ is a bridge}. \]
\[ R_{G_1 \sqcup G_2} = R_{G_1} \cdot R_{G_2}. \]

References