**Ribbon graphs**

**Definition.** A ribbon graph $G$ is a surface (possibly non-orientable) with boundary, represented as the union of two sets of closed topological discs called vertices $V(G)$ and edges $E(G)$, satisfying the following conditions:

- these vertices and edges intersect by disjoint line segments;
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.

\[
RG = x_{e}R_{G/e} + y_{e}R_{G-e}
\]

if $e$ is ordinary, that is neither a bridge nor a loop,

\[
RG = (x_{e} + Xy_{e})R_{G/e}
\]

if $e$ is a bridge.

\[
RG_{G_1 \sqcup G_2} = RG_1 \cdot RG_2
\]
Thistlethwaite’s Theorem [Ka1] Up to a sign and multiplication by a power of $t$ the Jones polynomial $J_L(t)$ of an alternating link $L$ is equal to the Tutte polynomial $T_T(-t, -t^{-1})$.

The theorem was generalized to non-alternating links using signed graphs in [Ka2] and using the Bollobás-Riordan polynomial for ribbon graphs in [DFKLS]; and to virtual links in [ChVo, Ch].

Theorem [Ch].

Let $L$ be a virtual link diagram with $e$ classical crossings, $G^*_L$ be the signed ribbon graph corresponding to a state $s$, and $v := v(G^*_L)$, $k := k(G^*_L)$. Then $e = e(G^*_L)$ and

$$[L](A, B, d) = A^e \left( X^k Y^v Z^{v+1} R_{G^*_L}(X, Y, Z) \right)_{\left| \begin{array}{c} X = \frac{Ad}{B} , \\ Y = \frac{Bd}{A} , \\ Z = \frac{1}{d} \end{array} \right. } .$$

Construction of a ribbon graph from a virtual link diagram

Diagram State $s$

Attaching planar bands Replacing bands by arrows
Untwisting state circles

Pulling state circles apart

Forming the ribbon graph $G^*_L$

References


