A Quantum Gauss-Bonnet Theorem

Tyler Friesen

November 13, 2014
Curvature in the plane

- Let $\Gamma$ be a smooth curve with orientation in $\mathbb{R}^2$, parametrized by arc length.
- The *curvature* $k$ of $\Gamma$ is $\pm \|\Gamma''\|$, where the sign is positive if $\Gamma''$ is counterclockwise of $\Gamma'$, and negative if $\Gamma''$ is clockwise of $\Gamma'$.
- Curvature measures the change in direction per unit distance along the curve.
Hopf’s Umlaufsatz

- A curve is *simple* if it has no self-intersections.
- Hopf’s Umlaufsatz: If $\Gamma$ is a simple smooth closed curve,
  \[ \int_{S^1} k(s) \, ds = \pm 2\pi \]
- Umlaufsatz with corners: If $\Gamma$ is a simple piecewise-smooth closed curve, let $C$ be the set of corners of $\Gamma$ and for each $c \in C$, let $\varphi_c$ be the exterior angle at $c$. Then
  \[ \int_{S^1} k(s) \, ds + \sum_{c \in C} \varphi_c = \pm 2\pi \]
The Whitney-Graustein Theorem

- Two smooth curves are said to be *regular homotopic* to each other if one can be continuously deformed into the other such that at any moment in time, the intermediate curve is a smooth curve.

- The *rotation number* of a smooth closed curve is given by the formula

  \[ \frac{1}{2\pi} \int_{S^1} k(s) \, ds \]

  It is always an integer, and this formula can be viewed as the Umlaufsatz with multiplicities.

- Whitney-Graustein Theorem: Two smooth closed curves in the plane are regular homotopic if and only if they have the same rotation number.
The $J^+$ invariant

- A curve is *generic* if its only self-intersections are transverse double points.
- Theorem: If two generic smooth closed curves in any surface are regular homotopic, then one can be deformed into the other by diffeomorphism of the surface and a finite number of self-tangency moves and triple-point moves.
- The $J^+$ invariant associates an integer to each generic smooth curve in a given orientable surface, such that this integer changes by 2 at direct self-tangency moves and is unchanged by opposite self-tangency moves and by triple-point moves.
- This almost determines $J^+$ uniquely; we just need to specify its value on a representative of each regular homotopy class. There is a standard such specification for planar curves.
The $J^+$ invariant (continued)

Figure: Self-tangency and triple-point moves

Figure: A direct self-tangency move and two opposite self-tangency moves

Images due to Lanzat and Polyak
Winding numbers

Given a curve $\Gamma$ in the plane and a point $p$ not in $\Gamma$, we define the *winding number* or *index* $\text{ind}_\Gamma(p)$ to be the total number of (signed) turns made by $\Gamma$ around $p$.

The winding number changes by $\pm 1$ when $p$ crosses over $\Gamma$, according to the orientation of the section of $\Gamma$ it crosses.

Given $\Gamma$, the winding number $\text{ind}_\Gamma$ thus gives a function on $\mathbb{R}^2 \setminus \Gamma$; we extend this to all of $\mathbb{R}^2$ by defining $\text{ind}_\Gamma(p)$ for $p \in \Gamma$ to be the average of the winding numbers of the regions of $\mathbb{R}^2 \setminus \Gamma$ in a neighborhood of $p$. 
Figures

Images due to Lanzat and Polyak
Lanzat and Polyak’s Polynomial Invariant

Let $\Gamma$ be a generic smooth closed curve in the plane. Let $X$ be its set of double points, and for each $d = \Gamma(t_1) = \Gamma(t_2) \in X$, let $\theta_d$ be the (non-oriented) angle between $\Gamma'(t_1)$ and $-\Gamma'(t_2)$.

Then Lanzat and Polyak define an associated “quantum invariant” $I_q(\Gamma) \in \mathbb{R}[q^{1/2}, q^{-1/2}]$ as follows:

$$
\frac{1}{2\pi} \left( \int_{S_1} k(t) \cdot q^{\text{ind}_\Gamma(\Gamma(t))} \, dt - \sum_{d \in X} \theta_d \cdot q^{\text{ind}_\Gamma(d)}(q^{1/2} - q^{-1/2}) \right)
$$

They showed using the Hopf Umlaufsatz with corners that the expression is invariant under planar isotopy.
Its relation to the rotation number and $J^+$

- Substituting $q = 1$ into Lanzat and Polyak’s polynomial gives
  \[
  \frac{1}{2\pi} \int_{S^1} k(s) \, ds,
  \]
  the rotation number. Hence we say it is a *quantum deformation* of the rotation number (or of the Umlaufsatz).

- Lanzat and Polyak showed that the first derivative of their polynomial at $q = 1$, $I_1' (\Gamma)$, changes by $-1$ under direct self-tangencies and is invariant under opposite self-tangencies and triple-point modifications, so $-2I_1' (\Gamma)$ changes by 2 at direct self-tangencies and is invariant under opposite self-tangencies and triple-point moves.

- Thus $J^+ (\Gamma) = -2I_1' (\Gamma)$ up to addition of some constant depending only on the regular homotopy class of $\Gamma$. 
Problems with generalizing to curves in surfaces

- What takes the role of the winding number in the definition of the integral?
- What replaces the Umlaufsatz in the proof?
- Lanzat and Polyak’s polynomial is a quantum deformation of the formula for rotation number; what should the generalization be a deformation of?
Homologically trivial curves

- The most important facts about the winding number are that it is locally constant on $S \setminus \Gamma$ and it changes by the appropriate amount when crossing over $\Gamma$.
- In some cases, there is no function on $S \setminus \Gamma$ satisfying this property:
  
  - When there is, we say $\Gamma$ is *homologically trivial*.
  - Given a homologically trivial curve $\Gamma$ in a connected oriented surface $S$ and $b \in S \setminus \Gamma$, let $\text{ind}_{\Gamma,b} : S \setminus \Gamma \to \mathbb{Z}$ be the unique locally constant function which sends $b$ to 0 and changes by the appropriate amount when crossing over $\Gamma$.
Geodesic Curvature

- To generalize the Umlaufsatz to surfaces we need a concept of curvature.
- In order to talk about curvature we need to have a concept of lengths and angles.
- The following concepts will apply to any surface with a Riemannian metric, but I will describe them for the less general case of a surface embedded in $\mathbb{R}^3$.
- As with planar curvature, we parametrize $\Gamma$ by arc length, but where planar curvature is the signed magnitude of $\Gamma''$, *geodesic curvature* is the signed magnitude of the projection of $\Gamma''$ onto the tangent plane $T_pS$.
- Examples: The geodesic curvature of a curve in the plane is its planar curvature. The geodesic curvature of a great circle on a sphere is constantly zero.
The Gauss-Bonnet Theorem

- Gauss-Bonnet Theorem: If $S$ is a closed subset of a surface and $\partial S$ is piecewise smooth with finite set $C$ of corners, then

$$\chi(S) = \frac{1}{2\pi} \left( \int_S K \, dA + \int_{\partial S} k_g \, ds + \sum_{c \in C} \varphi_c \right)$$

- If $S$ is the entire surface (without boundary) then this reduces to the Gauss-Bonnet theorem presented earlier by Dr. Farb.

- If $S$ is a subset of the plane and $S$ is homeomorphic to a disk, then this reduces to Hopf’s Umlaufsatz with corners.

- More generally, if $S$ is homeomorphic to a disk, the Gauss-Bonnet theorem is like the Umlaufsatz with a corrective term for Gaussian curvature.
Rotation numbers of homologically trivial curves

Given an oriented surface $S$, the *rotation number* is the unique way of assigning a value in $\mathbb{Z}/\chi(S)\mathbb{Z}$ (or $\mathbb{Z}$ if $\chi(S) = 0$) to each homologically trivial curve in $S$ such that

1. The rotation number is invariant under regular homotopies.
2. The rotation number of a small counterclockwise curve is 1.
3. The rotation number of the composition of two curves is the sum of their rotation numbers.
McIntyre and Cairns’s formula for the rotation number

- Pick a base point \(b\) in \(S \setminus \Gamma\).
- For \(j \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}\), let \(S_j\) be the region of \(S\) on which \(\text{ind}_{\Gamma,b}\) is greater than \(j\).
- For \(j \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}\), let
  \[
  a_j = \begin{cases} 
  \chi(S_j) - \chi(S) & j < 0 \\
  \chi(S_j) & j > 0 
  \end{cases}
  \]
- Then the winding number is given by
  \[
  \sum_{j \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}} a_j \mod |\chi(S)|
  \]
The Gauss-Bonnet Theorem with multiplicities

Let $\Gamma$ be a homologically trivial generic smooth curve in a connected closed surface $S$ with Riemannian metric and orientation.

We can calculate the Euler characteristics in McIntyre and Cairns’s formula using the Gauss-Bonnet Theorem; this gives the following formula for the rotation number:

$$\frac{1}{2\pi} \left( \int_{S^1} k_g(t) \, dt + \iint_S K \cdot \text{ind}_{\Gamma,b} \, dA \right) \mod |\chi(S)|$$

This can be viewed as the Gauss-Bonnet theorem with multiplicities.

Note that if we change $b$ so that $\text{ind}_{\Gamma,b}$ increases by 1 everywhere, the expression before taking the modulus changes by $\chi(S)$. 
The Quantum Gauss-Bonnet Theorem

Instead of taking $\sum_{j \in \mathbb{Z}/2\mathbb{Z}} a_j$, take $\sum_{j \in \mathbb{Z}/2\mathbb{Z}} a_j q^j$.

Again applying the Gauss-Bonnet Theorem to calculate the $a_j$'s, we get

$$\frac{1}{2\pi} \left( \int_{S^1} k_g(t) \cdot q^{\text{ind}_{\Gamma, b}(\Gamma(t))} \ dt \right)$$

$$+ \sum_{d \in X} (\pi - \theta_d) q^{\text{ind}_{\Gamma, b}(d)} (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) + \iint_S K \cdot \frac{q^{\text{ind}_{\Gamma, b}} - 1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \ dA$$

This is a topological invariant and a quantum deformation of the rotation number, but it isn’t quite a generalization of Lanzat and Polyak’s formula.
The Quantum Gauss-Bonnet Theorem (continued)

- The expression

\[
\frac{1}{2} \sum_{d \in X} (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) q^{\text{ind}_\Gamma,b}
\]

is a topological invariant and is equal to 0 at \( q = 1 \), so subtracting it away from the expression on the previous page will still give a topological invariant and deformation of the rotation number.

- Here it is:

\[
l_q(\Gamma, b) := \frac{1}{2\pi} \left( \int_{S^1} k_g(t) \cdot q^{\text{ind}_\Gamma,b}(\Gamma(t)) \, dt \right.
\]

\[
- \sum_{d \in X} \theta_d \cdot q^{\text{ind}_\Gamma,b}(d) \left( q^{\frac{1}{2}} - q^{-\frac{1}{2}} \right) + \iint_S K \cdot \frac{q^{\text{ind}_\Gamma,b} - 1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \, dA
\]
Its relation to $J^+$

- $l_q(\Gamma, b)$ changes the same way under self-tangency and triple-point moves that Lanzat and Polyak’s polynomial does, so we might think that $-2l_1'(\Gamma, b)$ gives us $J^+(\Gamma)$ (up to a constant depending on the regular homotopy class of $\Gamma$).

- However, $l_1'(\Gamma, b)$ is not invariant under a change of base point $b$.

- $l_1(\Gamma, b)$ (the formula for rotation number, before taking the modulus) can be used to produce a corrective term to give a formula which doesn’t change under change of base point.

- When $\chi(S) \neq 0$

$$J^+(\Gamma) = \frac{l_1(\Gamma, b)^2}{\chi(S)} - 2l_1'(\Gamma, b)$$

up to a constant depending on the regular homotopy class of $\Gamma$. 
An explicit formula for $J^+$

\[
\frac{1}{4\pi^2 \chi(S)} \left( \int_{S^1} k_g(t) \, dt + \iint_S \text{ind}_{\Gamma, b} \, dA \right)^2 \\
- \frac{1}{\pi} \left( \int_{S^1} k_g(t) \cdot \text{ind}_{\Gamma, b}(\Gamma(t)) \, dt - \sum_{d \in X} \theta_d + \frac{1}{2} \iint_S K \cdot (\text{ind}_{\Gamma, b})^2 \, dA \right)
\]