Graphs

Definition. A graph $G$ is a finite set of vertices $V(G)$ and a finite set $E(G)$ of unordered pairs $(x, y)$ of vertices $x, y \in V(G)$ called edges.

A graph may have loops $(x, x)$ and multiple edges when a pair $(x, y)$ appears in $E(G)$ several times. Pictorially we represent the vertices by points and edges by lines connecting the corresponding points. Topologically a graph is a 1-dimensional cell complex with $V(G)$ as the set of 0-cells and $E(G)$ as the set of 1-cells. Here are two pictures representing the same graph.

$$\begin{align*}
V(G) &= \{a, b, c, d\} \\
E(G) &= \{(a, a), (a, b), (a, c), (a, d), (b, c), (b, c), (b, d), (c, d)\}
\end{align*}$$

Tutte polynomial

Chromatic polynomial $C_G(q)$.

A coloring of $G$ with $q$ colors is a map $c : V(G) \to \{1, \ldots, q\}$. A coloring $c$ is proper if for any edge $e$: $c(v_1) \neq c(v_2)$, where $v_1$ and $v_2$ are the endpoints of $e$.

Definition 1. $C_G(q) := \#$ of proper colorings of $G$ in $q$ colors.

Properties (Definition 2).

$C_G = C_{G-e} - C_{G/e}$ ;
$C_{G_1 \sqcup G_2} = C_{G_1} \cdot C_{G_2}$, for a disjoint union $G_1 \sqcup G_2$ ;
$C_\ast = q$.

Tutte polynomial $T_G(x, y)$.

Definition 1.

$T_G = T_{G-e} + T_{G/e}$ if $e$ is neither a bridge nor a loop ;
$T_G = xT_{G/e}$ if $e$ is a bridge ;
$T_G = yT_{G-e}$ if $e$ is a loop ;
$T_{G_1 \sqcup G_2} = T_{G_1} \cdot T_{G_2}$ for a disjoint union $G_1 \sqcup G_2$
and a one-point join $G_1 \cdot G_2$ ;
$T_\ast = 1$.

Properties.

$T_G(1, 1)$ is the number of spanning trees of $G$ ;
$T_G(2, 1)$ is the number of spanning forests of $G$ ;
$T_G(1, 2)$ is the number of spanning connected subgraphs of $G$ ;
$T_G(2, 2) = 2^{|E(G)|}$ is the number of spanning subgraphs of $G$ .
$C_G(q) = q^{k(G)}(-1)^{r(G)}T_G(1-q, 0)$ .

Definition 2.

Let $\bullet$ $F$ be a graph;
$\bullet$ $v(F)$ be the number of its vertices;
$\bullet$ $e(F)$ be the number of its edges;
$\bullet$ $k(F)$ be the number of components of $F$;
\[ T_G(x, y) := \sum_{F \subseteq E(G)} (x - 1)^{r(G) - r(F)} (y - 1)^{n(F)} \]

Dichromatic polynomial \( Z_G(q, v) \) (Definition 3).

Let \( Col(G) \) denote the set of colorings of \( G \) with \( q \) colors.

\[ Z_G(q, v) := \sum_{c \in Col(G)} (1 + v) \# \text{ edges colored not properly by } c \]

\textbf{Properties}.

- \( Z_G = Z_{G-e} + vZ_{G/e} \);
- \( Z_{G_1 \sqcup G_2} = Z_{G_1} \cdot Z_{G_2} \), for a disjoint union \( G_1 \sqcup G_2 \);
- \( Z_* = q^k \);
- \( Z_G(q, v) = \sum_{F \subseteq E(G)} q^{k(F)} v^{e(F)} \);
- \( C_G(q) = Z_G(q, -1) \);
- \( Z_G(q, v) = q^{k(G)} v^{e(G)} T_G(1 + qv^{-1}, 1 + v) \);
- \( T_G(x, y) = (x - 1)^{-k(G)} (y - 1)^{-v(G)} Z_G((x - 1)(y - 1), y - 1) \).

\textbf{Potts model in statistical mechanics} (Definition 4).

Potts model (C.Domb 1952): \( q = 2 \) the Ising model (W.Lenz, 1920)

Let \( G \) be a graph.

Particles are located at vertices of \( G \). Each particle has a spin, which takes \( q \) different values. A state, \( \sigma \in S \), is an assignment of spins to all vertices of \( G \). Neighboring particles interact with each other only if their spins are the same.

The energy of the interaction along an edge \( e \) is \(-J_e \) (coupling constant). The model is called ferromagnetic if \( J_e > 0 \) and antiferromagnetic if \( J_e < 0 \).

Energy of a state \( \sigma \) (Hamiltonian),

\[ H(\sigma) = - \sum_{(a,b) \in E(G)} J_e \delta(\sigma(a), \sigma(b)). \]

Boltzmann weight of \( \sigma \):

\[ e^{-\beta H(\sigma)} = \prod_{(a,b) \in E(G)} e^{J_e \beta \delta(\sigma(a), \sigma(b))} = \prod_{(a,b) \in E(G)} \left( 1 + (e^{J_e \beta} - 1) \delta(\sigma(a), \sigma(b)) \right) \],

where the inverse temperature \( \beta = \frac{1}{\kappa T} \), \( T \) is the temperature, \( \kappa = 1.38 \times 10^{-23} \) joules/Kelvin is the Boltzmann constant.

The Potts partition function \( \text{for } x_e := e^{J_e \beta} - 1 \)

\[ Z_G(q, x_e) := \sum_{\sigma \in S} e^{-\beta H(\sigma)} = \sum_{\sigma \in S} \prod_{e \in E(G)} \left( 1 + x_e \delta(\sigma(a), \sigma(b)) \right) \]
**Properties of the Potts model**

Probability of a state $\sigma$: $P(\sigma) := e^{-\beta H(\sigma)} / Z_G$.

Expected value of a function $f(\sigma)$:

$$\langle f \rangle := \sum_{\sigma} f(\sigma) P(\sigma) = \sum_{\sigma} f(\sigma) e^{-\beta H(\sigma)} / Z_G .$$

Expected energy: $\langle H \rangle = \sum_{\sigma} H(\sigma) e^{-\beta H(\sigma)} / Z_G = -\frac{d}{d\beta} \ln Z_G$.

Fortuin–Kasteleyn’1972: $Z_G(q, x_e) = \sum_{F \subseteq E(G)} q^{k(F)} \prod_{e \in F} x_e ,$

where $k(F)$ is the number of connected components of the spanning subgraph $F$. $Z_G = Z_{G\setminus e} + x_e Z_{G/e}$.

**Spanning tree generating function (Definition 5).**

For a connected graph $G$ fix an order of its edges: $e_1, e_2, \ldots, e_m$. Let $T$ be a spanning tree.

An edge $e_i \in E(T)$ is called *internally active (live)* if $i < j$ for any edge $e_j$ connecting the two components of $T - e_i$.

An edge $e_j \not\in E(T)$ is called *externally active (live)* if $j < i$ for any edge $e_i$ in the unique cycle of $T \cup e_j$.

Let $i(T)$ and $j(T)$ be the numbers of internally and externally active edges correspondingly.

$$T_G(x, y) := \sum_T x^{i(T)} y^{j(T)}$$

**Doubly weighted Tutte polynomial.**

With each edge $e$ of a graph $G$ we associate two variables (weights) $u_e$ and $v_e$.

$$T_G(\{u_e, v_e\}; x, y) := \sum_{F \subseteq E(G)} \left( \prod_{e \in F} u_e \right) \left( \prod_{e \not\in F} v_e \right) (x - 1)^{r(G)-r(F)} (y - 1)^{n(F)}$$

Properties.

$$T_G = v_e T_{G-e} + u_e T_{G/e} \quad \text{if } e \text{ is neither a bridge nor a loop;}$$

$$T_G = (v_e (x - 1) + u_e) T_{G/e} \quad \text{if } e \text{ is a bridge;}$$

$$T_G = (v_e + (y - 1) u_e) T_{G-e} \quad \text{if } e \text{ is a loop;}$$

$$T_{G_1 \cup G_2} = T_{G_1} \cdot T_{G_2} \quad \text{for a disjoint union } G_1 \cup G_2$$

$$T_{G_1 \cdot G_2} = T_{G_1 \cdot G_2} \quad \text{and a one-point join } G_1 \cdot G_2 ;$$

$$T_* = 1 .$$

**Tutte polynomial of signed graphs.**

Signed graph is a graph with signs $\pm 1$ assigned to the edges of the graph.

We define the Tutte polynomial of a signed graph by substituting the following weights to the doubly weighted Tutte polynomial.

$+$-edge: $u_e := 1, \quad v_e := 1$; $-$-edge: $u_e := \sqrt{\frac{x - 1}{y - 1}}, \quad v_e := \sqrt{\frac{y - 1}{x - 1}}$. 
With this substitution the Tutte polynomial for signed graphs becomes

\[ T_G(x, y) = \sum_{F \subseteq E(G)} (x - 1)^{r(G) - r(F) + s(F) + s(F)} (y - 1)^{n(F) - s(F)}, \]

for \( s(F) := \frac{e_-(F) - e_-(E(G) \setminus F)}{2} \), where \( e_-(S) \) stands for the number of negative edges of \( S \).

**Chromatic polynomial of signed graphs.**

There are two chromatic polynomials of signed graphs.

A \( q \)-coloring of a signed \( G \) is a map \( c : V(G) \to \{-q, -q + 1, \ldots, -1, 0, 1, \ldots, q - 1, q\} \). A \( q \)-coloring \( c \) is proper if for any edge \( e \) with the sign \( \varepsilon_e \): \( c(v_1) \neq \varepsilon c(v_2) \), where \( v_1 \) and \( v_2 \) are the endpoints of \( e \).

**Definition.**

\( C_G(2q + 1) := \# \text{ of proper } q \text{-colorings of } G. \)

\( C_G^{\neq 0}(2q) := \# \text{ of proper } q \text{-colorings of } G \text{ which take nonzero values.} \)

**Properties.**

- \( C_G(\lambda) \) is a polynomial function of \( \lambda = 2q + 1 > 0 \);
- \( C_G^{\neq 0}(\lambda) \) is a polynomial function of \( \lambda = 2q > 0 \);
- \( C_G(\lambda) = C_G^{\neq 0}(\lambda) - C_G(\varepsilon) \);
- \( C_G^{\neq 0}(\lambda) = C_G^{\neq 0}(\lambda) - C_G^{\neq 0}(\varepsilon) \);
- \( C_{G_1 \sqcup G_2} = C_{G_1} \cdot C_{G_2} \) and \( C_{G_1 \sqcup G_2}^{\neq 0} = C_{G_1}^{\neq 0} \cdot C_{G_2}^{\neq 0} \) for a disjoint union \( G_1 \sqcup G_2 \);
- \( C_0 = 1 \).

**Problem.**

Express the chromatic polynomials of signed graphs in terms of the signed Tutte polynomial.