**Dual representable matroids**

Let \( E = \{v_1, \ldots, v_n\} \) be a collection of vectors in a vector space \( U \) and \( M \) be a matroid of their linear dependences. Consider an \( n \)-dimensional vector space \( V \) with a basis \( e_1, \ldots, e_n \) and a linear map \( f : V \to U \) sending \( e_k \) to \( v_k \). Denote the kernel of this map by \( W \). It is a subspace of \( V \) and there is a natural inclusion map \( i : W \hookrightarrow V \). There is the dual map \( W \overset{i^*}{\to} V^* \) of dual vector spaces. The space \( V^* \) has a natural dual basis \( e_1^*, \ldots, e_n^* \). Their images \( i^*(e_1^*), \ldots, i^*(e_n^*) \) is a collection of vectors in the space \( W^* \). These vectors with the structure of linear dependences between them form the dual matroid \( M^* \).

**∆-matroids** [Bouchet]

<table>
<thead>
<tr>
<th>Matroids</th>
<th>∆-matroids</th>
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<tbody>
<tr>
<td>A matroid is a pair ( M = (E, \mathcal{B}) ) consisting of a finite set ( E ) and a nonempty collection ( \mathcal{B} ) of its subsets, called bases, satisfying the axioms:</td>
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<tr>
<td>(B1) No proper subset of a base is a base.</td>
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<td>(B2) Exchange axion If ( B_1 ) and ( B_2 ) are bases and ( b_1 \in B_1 - B_2 ), then there is an element ( b_2 \in B_2 - B_1 ) such that ( (B_1 - b_1) \cup b_2 ) is a base.</td>
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<tr>
<td>A ∆-matroid is a pair ( M = (E, \mathcal{F}) ) consisting of a finite set ( E ) and a nonempty collection ( \mathcal{F} ) of its subsets, called feasible sets, satisfying the Symmetric Exchange axion</td>
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<tr>
<td>If ( F_1 ) and ( F_2 ) are two feasible sets and ( f_1 \in F_1 \Delta F_2 ), then there is an element ( f_2 \in F_1 \Delta F_2 ) such that ( F_1 \Delta {f_1, f_2} ) is a feasible set.</td>
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**Ribbon graphs (graphs on surfaces)**

**Definition.** A ribbon graph \( G \) is a surface (possibly non-orientable) with boundary, represented as the union of two sets of closed topological discs called vertices \( V(G) \) and edges \( E(G) \), satisfying the following conditions:

- these vertices and edges intersect by disjoint line segments;
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.

**Definition.** A quasi-tree is ribbon graph \( G \) with a single boundary component, \( bc(G) = 1 \).

**Examples.**

Spanning quasi-trees: \( \{a\}, \{b\}, \{a, b, c\} \)

Spanning quasi-trees: \( \emptyset, \{a, b\} \)

Spanning quasi-trees: \( \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\} \)

**Theorem.** Let \( G = (V, E) \) be a ribbon graph. Then \( D(G) := (E, \{\text{spanning quasi-trees}\}) \) is a ∆-matroid.
Minors in $\Delta$-matroids

Let $D = (E, F)$ be a $\Delta$-matroid and $e \in E$. $e$ is a loop iff $\forall F \in \mathcal{F}, e \not\in F$. $e$ is a coloop iff $\forall F \in \mathcal{F}, e \in F$.

If $e$ is not a loop, $D/e := (E \setminus \{e\}, \{F \setminus \{e\}|F \in \mathcal{F}, e \in F\})$.

If $e$ is not a coloop, $D \setminus e := (E \setminus \{e\}, \{F \setminus \{e\}|F \in \mathcal{F}, F \subset E \setminus \{e\}\}$.

Twists of $\Delta$-matroids. Let $D = (E, F)$ be a $\Delta$-matroids and $A \subset E$.

$D * A := (E, \{F \Delta A|F \in \mathcal{F}\})$.

Dual $\Delta$-matroid: $D^* := D + E$.

Matroids associated with a $\Delta$-matroid

Let $D = (E, F)$ be a $\Delta$-matroid.

$D_{\min} := (E, F_{\min})$, where $F_{\min} := \{F \in \mathcal{F}|F$ is of minimal possible cardinality\}.

$D_{\max} := (E, F_{\max})$, where $F_{\max} := \{F \in \mathcal{F}|F$ is of maximal possible cardinality\}.

Facts.

- $D_{\min}$ and $D_{\max}$ are usual matroids. Width $w(D) := r(D_{\max}) - r(D_{\min})$.
- $(D(G))_{\min} = C(G)$. $(D(G))_{\max} = (C(G^*))^*$.
- $D(G) = C(G)$ iff $G$ is a planar ribbon graph.

Matroid perspectives (M. Las Vergnas [LV])

Definition. Let $M$ and $M'$ be two matroid structures on the same ground set $E$. They form a matroid perspective $M \rightarrow M'$ if any circuit of $M$ is a union of circuits of $M'$. Equivalently, $r_M(X) - r_M(Y) \geq r_{M'}(X) - r_{M'}(Y)$ for all $Y \subseteq X \subseteq E$.

Example.

For graphs $G$ and $G^*$ dually embedded in a surface, then the bond matroid of $G^*$ and the circuit matroid of $G$ form a matroid perspective, $B(G^*) \rightarrow C(G)$.

Lemma. For any $\Delta$-matroid $D$, $D_{\max} \rightarrow D_{\min}$ is a matroid perspective.

Tutte like polynomials

The Las Vergnas polynomial ([LV]) of a matroid perspective $M \rightarrow M'$.

$$T_{M \rightarrow M'}(x, y, z) := \sum_{X \subseteq M} (x - 1)^{r_{M'}(E) - r_{M'}(X)}(y - 1)^{r_M(X)}z^{r_M(E) - r_{M'}(E) - r_{M'}(X)}$$

Properties. $T_M(x, y) = T_{M \rightarrow M}(x, y, z)$; $T_{M'}(x, y) = (y - 1)^{r_{M'}(E) - r_{M'}(X)}T_{M \rightarrow M'}(x, y, \frac{1}{y-1})$.

The Bollobás-Riordan polynomial ([BR]) of a ribbon graph $G$.

$$R_G(X, Y, Z) := \sum_{F \subseteq G} X^{r(G) - r(F)} Y^{n(F)} Z^{k(F) - bc(F) + n(F)}$$

References

