Joseph Paugh, Justin Wu, Gavin Zhang

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Main Result

The Arrow Polynomial

A Polynomial Invariant of Virtual Link Diagrams

Joseph Paugh, Justin Wu, Gavin Zhang

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Outline of Presentation

Joseph Paugh,
Justin Wu,
Gavin Zhang

Motivation

Joseph Paugh,
Justin Wu,
Gavin Zhang

- There are many virtual knots that are not distinguished by the Jones polynomial.
- For example, both the Kishino Knot (pictured below) and the unknot have unit Jones polynomials.
- We also want to determine whether a virtual link diagram is actually classical.
- Can we construct an invariant that will tell us when a virtual link has virtual crossing number zero?

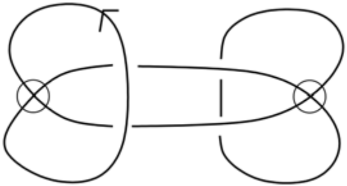
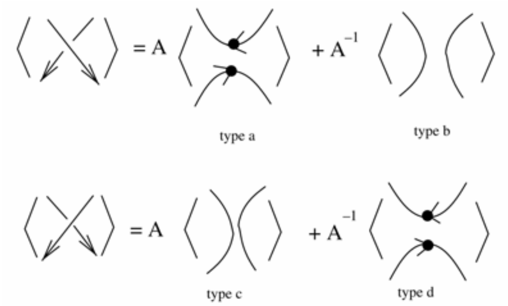


Diagram of the Kishino Knot

States of the Arrow Polynomial

- Given a virtual link diagram, a state is obtained according to the following oriented state expansion:



- The arrows resulting from horizontal splittings are decorated vertices; they do not indicate orientation.
- Except for the arrows, these are identical to the A and B type smoothings for the Jones Polynomial.

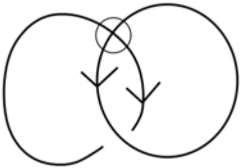
Defining the Arrow Polynomial

- If two adjacent nodes are decorated with arrows pointing in the same direction along the loop, they "cancel."

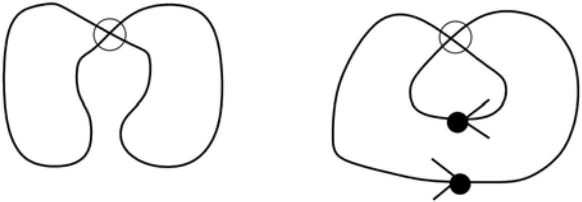
Reduction of Arrows

- After reducing every such pair of arrows on a loop, we call the result a reduced loop.

Examples of States

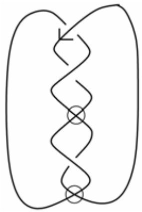


The Virtual Hopf Link

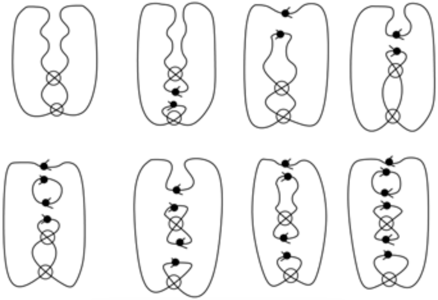


States of The Virtual Hopf Link

Examples of States



The Virtualized Trefoil



States of The Virtualized Trefoil

Evaluating States

- Each reduced loop C has an even number of nodes $2n$.
- Define $\langle C \rangle = 1$ if $n = 0$ and $\langle C \rangle = K_n$ if $n > 0$, where K_1, K_2, ... are independent commuting variables.
- Then define $\langle S \rangle = \prod_C \langle C \rangle$.

The Arrow Polynomial

Definition

State Sum of an Oriented Virtual Link. The arrow polynomial $\langle K \rangle_A$ of an oriented virtual link K is the polynomial in $\mathbb{Z}[A, A^{-1}, K_1, K_2, \dots]$ defined by

$$\langle K \rangle_A = \sum_S A^{\alpha-\beta} d^{|S|-1} \langle S \rangle,$$

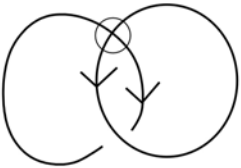
where:

- α is the number of smoothings in S with coefficient A;
- β is the number of smoothings in S with coefficient A^{-1};
- $d := -A^2 - A^{-2}$;
- $|S|$ is the number of loops in S.

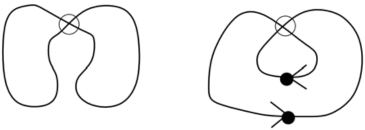
Example: The Virtual Hopf Link

- Call the left state S_1 and the right state S_2. Then $\langle S_1 \rangle = 1$, $\langle S_2 \rangle = K_1$, and $|S_1| = |S_2| = 1$.
- Therefore, the arrow polynomial of this link is

$$\begin{aligned} \langle VHL \rangle_A &= A^{-1} d^0 + A d^0 K_1 \\ &= A^{-1} + K_1 A \end{aligned}$$

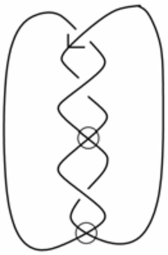


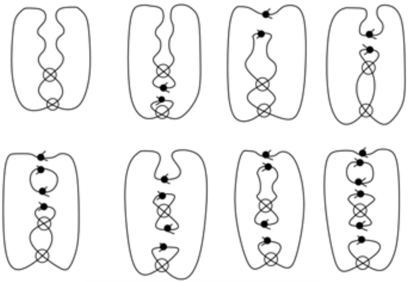
The Virtual Hopf Link



States

Example: The Virtualized Trefoil





- Labeled $1-8$ from left-to-right and top-to-bottom, the summands contributed by each of these eight states are:

State 1: $A^1d^1 = -A^3 - A^{-1}$
State 2: $A^3d^0 = A^3$
State 3: $A^{-1}d^0 = A^{-1}$
State 4: $A^{-1}d^0 = A^{-1}$
State 5: $A^{-3}d^1 = -A^{-5} - A^{-1}$
State 6: $A^1d^1K_1^2 = (-A^3 - A^{-1})K_1^2$
State 7: $A^1d^1K_1^2 = (-A^3 - A^{-1})K_1^2$
State 8: $A^{-1}d^2K_1^2 = (A^3 + A^{-5} + 2A^{-1})K_1^2$

- Therefore, the virtualized trefoil has arrow polynomial $\langle VT\rangle_A = -A^{-5} + A^{-5}K_1^2 - A^3K_1^2$.

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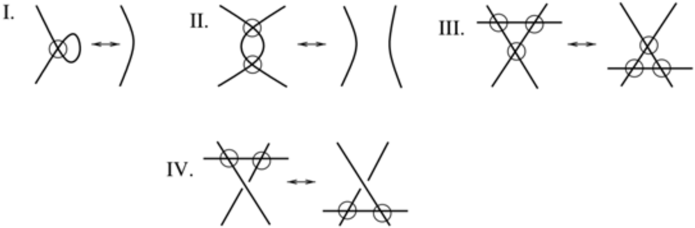
Theorem

Let K be a virtual link diagram. The polynomial $\langle K \rangle_A$ is invariant under the Reidemeister moves II and III and virtual Reidemeister moves.

Virtual Reidemeister Move Invariance

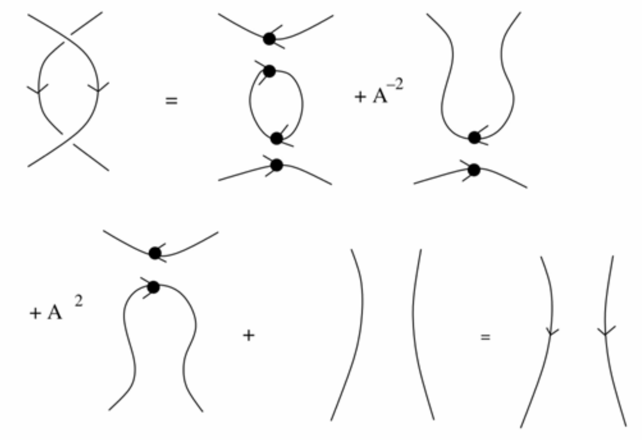


Virtual Reidemeister Moves

- Invariance under the Virtual Reidemeister Moves I-III is obvious: these moves only involve virtual crossings.
- Invariance under the Virtual Reidemeister IV move can be seen by applying the skein relation and the Virtual Reidemeister IV move.

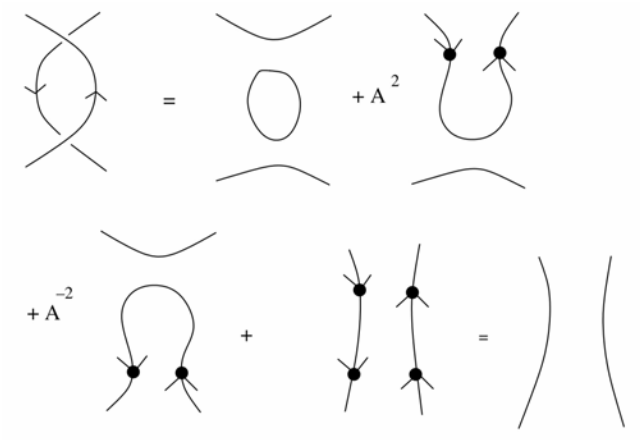
Reidemeister II Invariance

- Reidemeister II invariance follows from direct computation:



Reidemeister II, Type 1

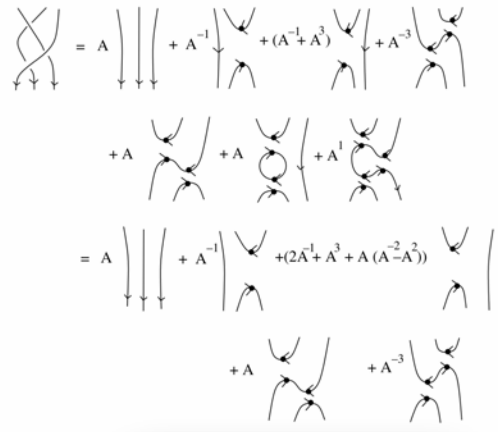
Reidemeister II Invariance



Reidemeister II, Type 2

Reidemeister III Invariance

- To prove invariance, we independently reduce the two diagrams related by the Reidemeister III move:



Reidemeister III Left Side

Reidemeister III Invariance

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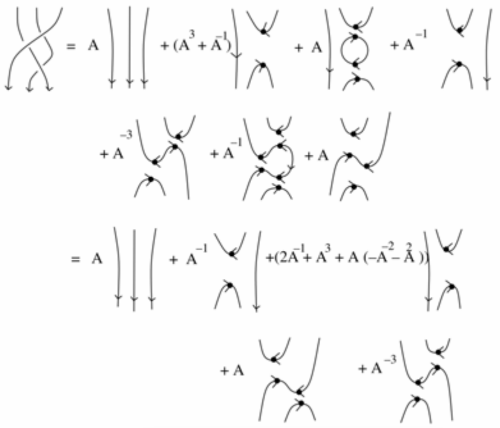

Reidemeister III Right Side

- Note that both diagrams yield the same states.

What about Reidemeister I Moves?

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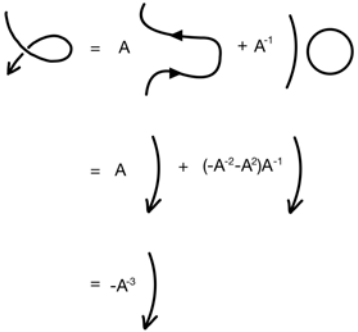
- A Reidemeister I move changes $\langle K \rangle_A$ by a factor of $-A^{-3}$ or $-A^3$, according to the sign of the crossing.



Reidemeister I Calculation on a Negative Crossing

The Normalized Arrow Polynomial

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Justin Wu,
Gavin Zhang

- By normalizing the arrow polynomial, we obtain invariance under Reidemeister I moves:

Definition

The (Normalized) Arrow Polynomial. The normalized arrow polynomial of an oriented virtual link K is the polynomial $\langle K \rangle_{NA} \in \mathbb{Z}[A, A^{-1}, K_1, K_2, \dots]$ defined by

$$\langle K \rangle_{NA} = (-A^3)^{-w(K)} \langle K \rangle_A,$$

where $w(K)$ denotes the writhe of K.

- By the previous theorem, $\langle K \rangle_{NA}$ is invariant under the virtual and classical Reidemeister moves.

Normalized Virtual Hopf Link

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- Recall that the Virtual Hopf Link (VHL) has arrow polynomial
$$\langle VHL \rangle_A = A^{-1} + K_1 A.$$
- Given that $w(VHL) = -1$, the normalized arrow polynomial for the Virtual Hopf Link is
$$\langle VHL \rangle_{NA} = -A^3(A^{-1} + K_1 A).$$
- Likewise, the Virtualized Trefoil (VT) has arrow polynomial
$$\langle VT \rangle_A = -A^{-5} + A^{-5}K_1^2 - A^3 K_1^2.$$
- Since $w(VT) = 1$,
$$\langle VT \rangle_{NA} = -A^{-3}(-A^{-5} + A^{-5}K_1^2 - A^3 K_1^2).$$

The Arrow Number and Virtual Crossings

Joseph Paugh,
Justin Wu,
Gavin Zhang

- Using the arrow polynomial, we obtain a lower bound on the virtual crossing number of a virtual link diagram. First, we need the following definition:

Definition

The *k-degree* of the product $A^m(K_{i_1}^{j_1} K_{i_2}^{j_2} \dots K_{i_v}^{j_v})$ is defined to be

$$i_1 \times j_1 + i_2 \times j_2 + \cdots + i_v \times j_v.$$

- The k-degree of a summand of $\langle K \rangle_A$ equals half the number of vertices in the reduced state associated to the summand.
- Notation: $AS(K)$ denotes the set of k-degrees obtained from summands of $\langle K \rangle_A$.

Main Result

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Justin Wu,
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Theorem (Dye, Kauffman)

The Arrow Polynomial of Classical Links. If K is a classical link diagram, then $AS(K) = \{0\}$.

- More generally, we have the following important result:

Theorem (Dye, Kauffman)

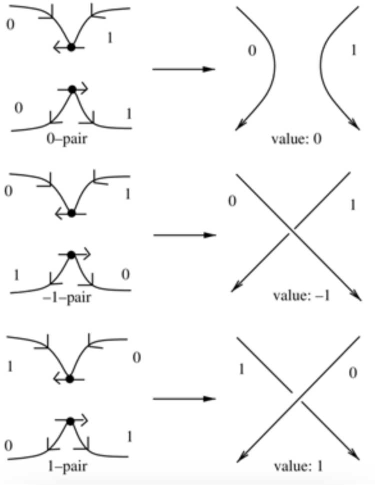
Let K be a virtual link diagram. Then the virtual crossing number of K, $v(K)$, is greater than or equal to the maximum k-degree of $\langle K \rangle_A$.

Proof Idea

- Given a state S of the arrow polynomial of K, we construct a classical link diagram $\lambda(S)$.
- Start with an alternating $0 - 1$ edge labeling of S.
- Convert virtual crossings to classical crossings: edges labeled 1 will always pass over edges labeled 0, and otherwise the strand that passes from left to right (in the direction of the diagram) will be the overcrossing strand.

Proof Idea

- We resolve each horizontal smoothing according to the right-hand diagram.
- If a crossing results, the 0 strand always passes over the 1 strand.



Proof Idea

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The resulting link diagram $\lambda(S)$ has the following properties:

- The sum of crossing signs where a 1 strand passes over a 0 strand never exceeds the number of virtual crossings in K.
- The sum of crossing signs where a 0 strand passes over a 1 strand equals the k-degree of S for some labeling.
- The latter two sums are equal.
- This applies to the representation of K with the least number of virtual crossings $v(K)$, and to any state S whose k-degree is $\max\{AS(K)\}$; the theorem follows.

Example

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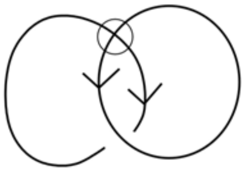
Justin Wu,
Gavin Zhang

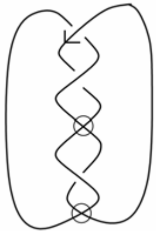
- The Virtual Hopf Link (VHL) has arrow polynomial $A^{-1} + K_1 A$.
- $AS(VHL) = \{0, 1\}$, so $v(VHL) \geqslant 1$.
- We have a diagram of the Virtual Hopf Link with one crossing:



hence $v(VHL) \leqslant 1$.
- Thus $v(VHL) = 1$, so the Virtual Hopf Link is not classical.

Example

- The Virtualized Trefoil (VT) has arrow polynomial $-A^{-5} + A^{-5}K_1^2 - A^3K_1^2$.
- Thus $AS(VT) = \{0, 2\}$ and $v(VT) \geqslant 2$.
- We have a diagram of the Virtualized Trefoil with two crossings:



  hence $v(VT) \leqslant 2$.
- Thus $v(VT) = 2$, so the Virtualized Trefoil is not classical.

References

H. A. Dye, L. H. Kauffman, Virtual crossing number and the arrow polynomial, J. Knot Theory Ramifications, 18(10)(2009)1335-1357.