Counting Acyclic Orientations of Signed Graphs

A Generalization of Stanley's Symmetric Acyclicity Theorem

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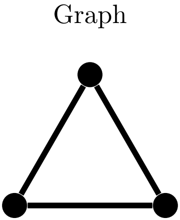
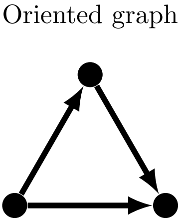
Oriented Graphs

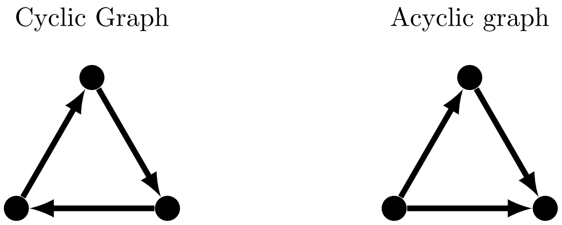
An orientation of a graph is a way of assigning an arrow to each edge.





Cycles

An oriented graph is called cyclic if it is possible to start at some vertex and follow the arrows until you end up where you started. Such a path is called a *cycle*.



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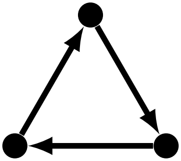
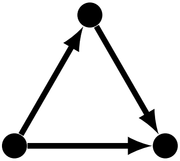
Cyclic Graph



Acyclic graph



Every acyclic oriented graph must have a vertex which has only arrows going into it, called a *sink*, and a vertex which has only arrows coming out of it, called a *source*.

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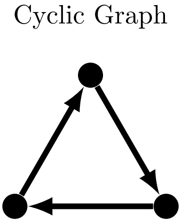
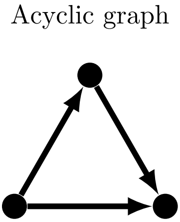


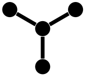


Every acyclic oriented graph must have a vertex which has only arrows going into it, called a *sink*, and a vertex which has only arrows coming out of it, called a *source*.

Another way of defining a cycle is as a closed path which contains no sinks or sources.

Counting Acyclicities

Question: Given a graph G, how many acyclic orientations of G have a certain number of sinks?

For example, consider the graph $G =$ 

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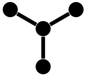
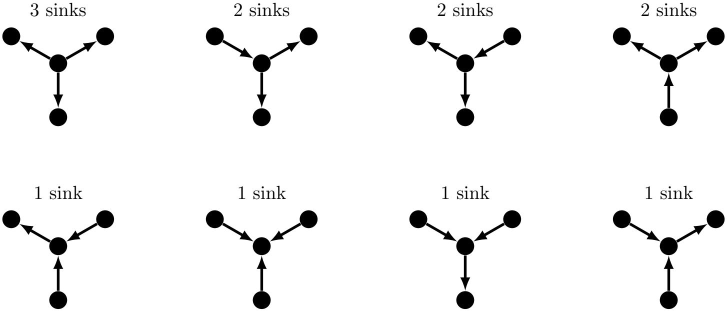
For example, consider the graph $G =$ 



G has 1 orientation with 3 sinks, 3 orientations with 2 sinks and 4 orientations with 1 sink.

Symmetric Chromatic Function

In 1995, Richard Stanley introduced the *Symmetric Chromatic Function*, X_G, of a graph G as a tool for investigating properties of graphs.

$$X_G(x_1, x_2, \dots) = \sum_{\substack{\kappa \text{ is a proper} \\ \text{coloring of } G}} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}$$

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The *elementary symmetric functions* are

$e_n = \sum_{0 < i_1 < \dots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$ and so $X_G = 2e_2$

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Theorem (Stanley)

When X_G is expressed in the elementary basis, the sum of the coefficients of the terms with k factors, is equal to the number of acyclic orientations of G with k sinks.

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X  $= 5e_1e_3 + e_1^2e_2 - 2e_2^2 + 4e_4$

$$= (4e_4) + (5e_1e_3 - 2e_2e_2) + (e_1e_1e_2)$$

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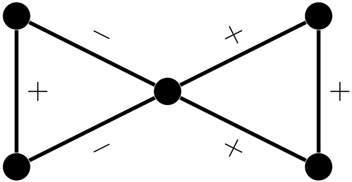
So there are 4 acyclic orientations with 1 sink, $5 - 2 = 3$ with 2 sinks, and 1 with 3 sinks.

The Land of Signed Graphs

A signed graph is a graph where each edge has either a plus sign or a minus sign. The sign of an edge e is $\sigma(e)$.



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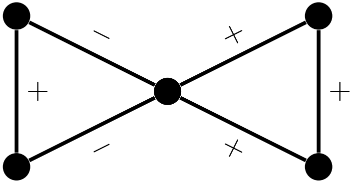

Signed edges have special interactions with graph colorings and orientations.

Colorings

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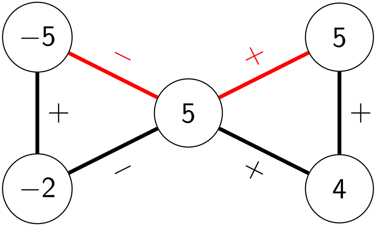
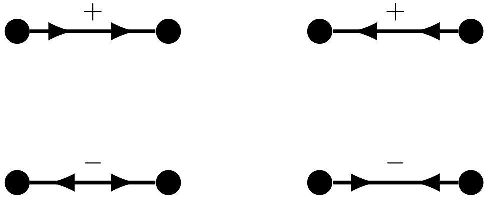
Two vertices u, v are improperly colored if there is an edge e connecting them and $\kappa(u) = \sigma(e)\kappa(v)$.



A coloring is proper if no vertices are improperly colored.

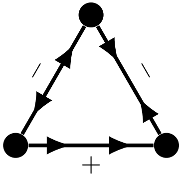
Orientations

Edges can be oriented differently according to the sign of the edge.



Acyclicity

A signed graph has a cycle if there is a closed path which contains no sinks or sources.



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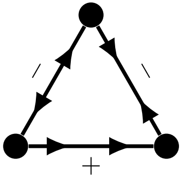

So, a signed graph is acyclic iff every closed path contains a sink or a source vertex.

B-Symmetric Chromatic Function

In analogy with Stanley's work, we can define the *B-Symmetric Chromatic Function* X_Σ of a signed graph Σ, in the variables $\dots, x_{-1}, x_0, x_1, \dots$

$$X_\Sigma(\dots, x_{-1}, x_0, x_1, \dots) = \sum_{\substack{\kappa \text{ is a proper} \\ \text{coloring of } \Sigma}} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}$$

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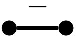
If Σ is the signed graph , then

$$X_\Sigma = \left(\sum_{i\in\mathbb{Z}} x_i\right)^2 - \sum_{i\in\mathbb{Z}} x_i x_{-i}$$

Generalization

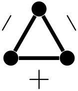
Let $e_n(\ldots, x_{-1}, x_0, x_1, \ldots) = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \ldots x_{i_n}$ for $n \in \mathbb{N}$.

Theorem

Then there exists families of functions $\{q_{a,b} \mid a, b \geq 1\}$ and $\{z_n \mid n = 0, 1, \ldots\}$ such that any X_Σ can be uniquely expressed in terms of sums and products of elements from $\{e_n \mid n = 1, 2, \ldots\} \cup \{q_{a,b} \mid a, b \geq 1\} \cup \{z_n \mid n = 0, 1, \ldots\}$ and when written this way, the sum of the coefficients of the all of the terms of the form $\left(\prod q_{a,b} \cdot \prod z_n\right) \cdot e_{n_1} \cdots e_{n_k}$, is the number of acyclic orientations of Σ with k sinks.

Example

For example, let Σ be the signed graph



Then

$$X_\Sigma = 2e_1e_2 + 2q_{1,1}e_1 + 2q_{2,1}$$

And so Σ has 2 acyclic orientation with 2 sinks, 2 acyclic orientations with 1 sink and 2 acyclic orientations with 0 sinks.

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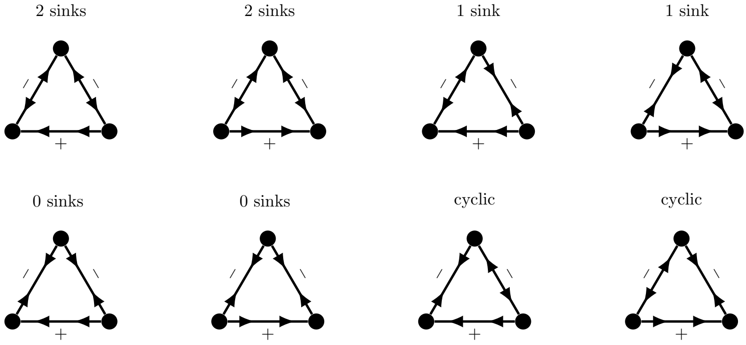

So there are 2 acyclic orientations with 2 sinks, 2 with 1 sink and 2 with no sinks.

Greatly abbreviated proof sketch

Each term of X_Σ corresponds to a coloring of Σ. We group terms of X_Σ according to which orientation the colorings correspond to. Our secret weapon is a linear and multiplicative function φ which sends the sum over all colorings corresponding to an orientation P to $t^{(\text{number of sinks of } P)}$. Hence φ sends X_Σ to a polynomial where the coefficients are the number of orientations with k sinks.

$$\varphi(X_\Sigma) = \sum_{\text{orientations } P} t^{\text{number of sinks of } P}$$

$$= \sum_{k=0}^{\infty} (\text{number of orientations with } k \text{ sinks}) \cdot t^k$$

Largely truncated proof sketch

It is a lot of trouble to show that φ is well-defined and works, but after this it's straightforward to show that φ sends each element of the e-basis to t, and each other extra basis terms to 1. This gives the statement of the theorem.

$$\begin{aligned}
\varphi(X_\Sigma) &= \varphi\left(\sum z_m \cdot q_{i_1,i_2} \cdots q_{i_{j-1},i_j} \cdot e_{n_1} \cdots e_{n_k}\right) \\
&= \sum \varphi(z_m) \cdot \varphi(q_{i_1,i_2}) \cdots \varphi(q_{i_{l-1},i_l}) \cdot \varphi(e_{n_1}) \cdots \varphi(e_{n_k}) \\
&= \sum 1 \cdot 1 \cdots 1 \cdot \underbrace{t \cdots t}_{k \text{ times}} \\
&= \sum_{k=0}^{\infty} (\text{number of terms with } k \text{ `e' factors}) \cdot t^k
\end{aligned}$$

Acknowledgements

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- Jake Huryn
- The Knots & Graphs research program at OSU
- The audience

Sources

- R. Stanley, A symmetric function generalization of the chromatic polynomial of a graph, Advances in Math. 111(1) (1995) 166–194.
- T. Zaslavsky, Signed graph coloring, Discrete Mathematics 39(2) (1982) 215–228.