Cut Points & Checkerboard Framings

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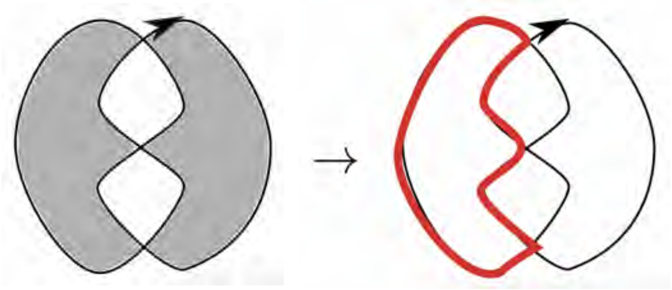
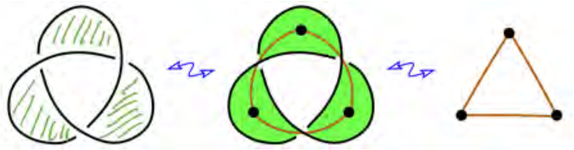
Motivation

- A checkerboard coloring is equivalent to an edge labeling in two colors, with the property that
  1. A color is assigned to each edge.
  2. The colors respect crossings: the left- and right-hand side of crossings have distinct color assignments.



Motivation

- Every classical knot has a checkerboard coloring.
- This has applications in connecting knot theory to graph theory, namely through Thistlethwaite's Theorem.



Thistlethwaite's Theorem

Motivation

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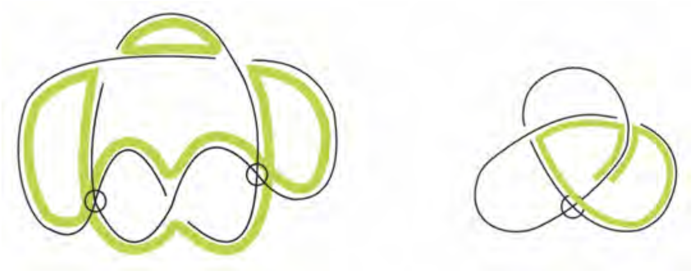
- Applying this to virtual links, we find that some virtual links have such an edge coloring, while others do not.
- Is it possible to extend this notion of a checkerboard coloring to *all* virtual links?



Cut Points

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- We assign vertices (”cut points”) to the edges of a link diagram to make it ”checkerboard colorable”
- Cut Points act as an artificial crossing, where coloring assignments must alternate across the cut point.

Checkerboard Framings

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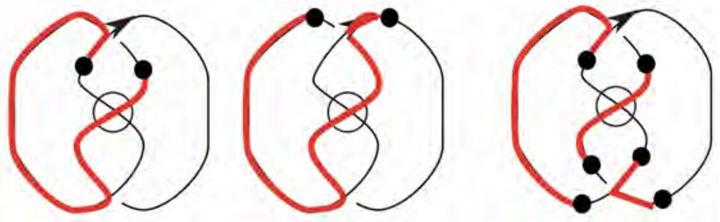
- *Checkerboard* framing : An assignment of cut points and colors to the edges of a virtual link diagram such that:
  1. Each edge is assigned a color.
  2. Edge colors alternate.
  3. Edge colors respect crossings.



Cut Point Number

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- Given a virtual link diagram D with framing F, let $\mathcal{P}(D, F)$ denote the number of cut points in F.

Theorem

1. $\mathcal{P}(D, F)$ *is even.*
2. $0 \leqslant \mathcal{P}(D, F) \leqslant 2n$; *$n = \#$ of classical crossings in D.*

- The *cut point number* $\mathcal{P}(D)$ is the minimum number of cut points required to frame any diagram equivalent to D:

$$\mathcal{P}(D) = \min\{\mathcal{P}((K, F)) \mid K \sim D \text{ and } F \text{ is a framing of } K\}.$$

Proof that $\mathcal{P}(D, F)$ is even

- Let (D, F) be a checkerboard framing with n crossings. Then, D has $2n$ edges.
- If D is a knot, the checkerboard framing alternates between colors. For the first edge and the last edge to have different colors, D must have an even number of cut points.
- If D is a link diagram, it is possible for a component to share crossings with other components. This means that the component may have an odd number of edges.
- If the edges of the framed component alternate, then the component has an odd number of cut points. However, components of this type occur in pairs so that the total number of cut points is even.

Cut Point Invariance

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- By definition:

Theorem

$\mathcal{P}(K)$ is a Virtual Link invariant.

- As a corollary, we obtain:

Corollary

If $\mathcal{P}(D) > 0$, then D is not a classical link.

Example: A False Converse

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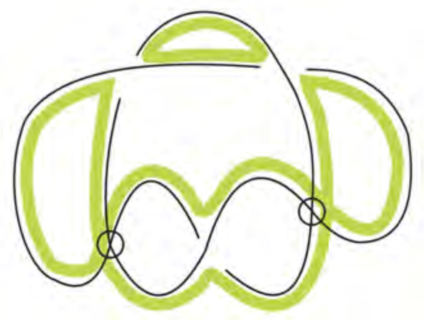
- It is not true that $\mathcal{P}(D) = 0$ if and only if D is classical.
- Counterexample:



- This diagram D is not classical, but $\mathcal{P}(D) = 0$.

Example: Minimum Number of Cut Points

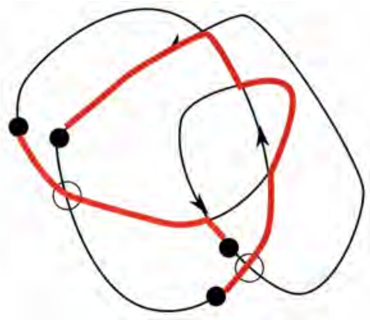
- The Miyazawa Knot L has a framing with two cut points:

(a) Minimum number of cut points

- Hence $\mathcal{P}(L) \leqslant 2$.

Example: A Strategy for Framing Links

- By adding pairs of cut points, virtual crossings can be made to behave like classical crossings in a framing.



Example: A Strategy for Framing Links

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- Since every classical knot is checkerboard colorable, every virtual link diagram can be framed in this way. Therefore:

Theorem

$$\mathcal{P}_d(D) \leqslant 2v_d(D),$$

where:

- $v_d(D) = \#$ *of virtual crossings in* D.
- $\mathcal{P}_d(D) =$ *minimum* $\#$ *of cut points required to frame* D.

Example: A Strategy for Framing Links

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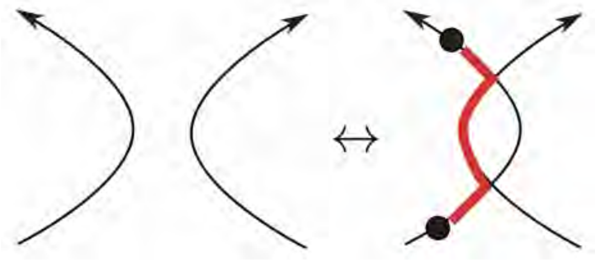
Corollary

$$\mathcal{P}(D) \leqslant 2v(D)$$

- In particular, $v(D) = 0 \implies \mathcal{P}(D) = 0$; this is another way to see that $\mathcal{P}(D) > 0$ implies D is not classical.

Checkerboard Framings and Reidemeister Moves

- Checkerboard framings are *not* invariant under the Reidemeister moves.
- Allowing for multiple cut points on each edge, the only Reidemeister move that affects the number of cut points is a Reidemeister Type II move:



Cut Point Moves

- A checkerboard framing of a virtual link diagram D can be modified using the following cut points moves:

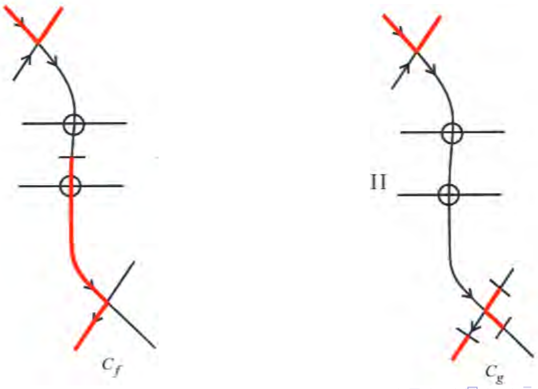
(a) Move I (b) Move II

- Move I can create a situation where an edge of D contains two cut points.
- The number of cut points on an edge can be reduced to 0 or 1 using Move II, resulting in a checkerboard framing.

Main Result

Theorem

Any two checkerboard framings of a diagram are related by a sequence of the cut point moves



From C_f to C_g

Main Result: Proof

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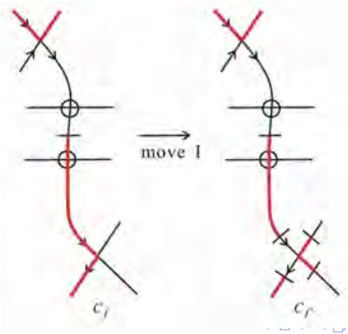
- Suppose f and g are framings of a diagram D.
- For each classical crossing in D, there are two possible colorings of adjacent edges that respect the crossing.
- Apply cut point move I to f at each crossing where f and g differ; call the result f'.
- Then f' and g have the same coloring at each crossing.



Main Result: Proof

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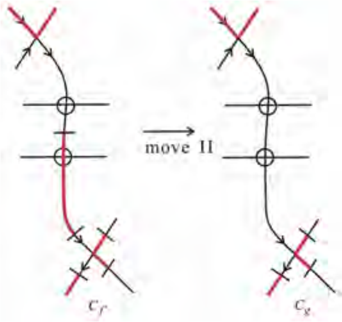
- f' and g may be different in edges whose two endpoints are cut points.
- Apply cut point move II to f' until each edge has at most one cut point; the result is g.



Our Project

- Project goal: Constructing a ribbon graph from a virtual link diagram with cut points so that the the Bollóbas–Riordan polynomial of the graph would be specialized to the arrow polynomial.

Reminder: The Bollobas-Riordan Polynomial

Definition

$R_G(x, y, z) = \sum_{F \in F(G)} x^{r(G)-r(F)} y^{n(F)} z^{k(F)-bc(F)+n(F)}$

- $k(G)$ denotes connected components
- $r(G)$ denotes rank of G
- $n(G)$ denotes nullity of G
- $bc(G)$ denotes number of connected components of boundary of G

Example: Alternating Virtual Links

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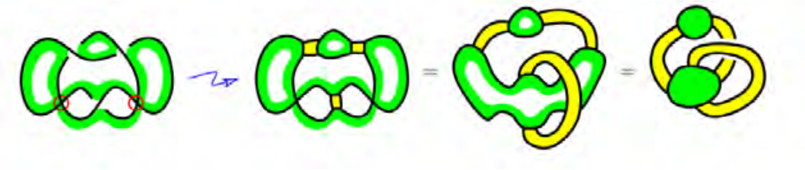
- Given a checkerboard colored alternating virtual link L, we can construct a ribbon graph G_L as follows:



Connection of Polynomials

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- Using this ribbon graph construction, we obtain a Thistlethwaite-type theorem for alternating virtual links:

Theorem

Let L be an alternating virtual link diagram and G_L be the corresponding ribbon graph. Then

- $[L](A, B, d) = A^{r(G)} B^{n(G)} d^{k(G)-1} R_{G(L)}(\frac{Bd}{A}, \frac{Ad}{B}, \frac{1}{d})$

Proof of the Theorem

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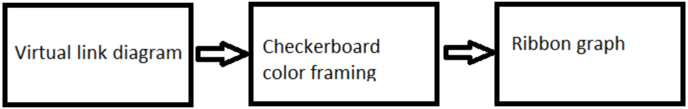
1. Crossings of the diagram L correspond to the edges of $G = G_L$, and there exists a one to one mapping φ from states of L to spanning subgraphs of G.
   - A-splitting means we keep the edge in our spanning subgraph, B-splitting removes the edge.
2. For all $F = \varphi(S)$, $e(F) = \alpha(S)$ and $e(G) - e(F) = \beta(S)$.
3. Substitute: $x = \frac{Bd}{A}$, $y = \frac{Ad}{B}$, $z = \frac{1}{d}$.
4. Algebra shows that the corresponding state and subgraph contribute equal terms to $[L]$ and R_G respectively.

Goal of Project

- As in the above theorem, we want to construct a ribbon graph from a checkerboard framing.



- Using this construction, we want to produce a Thistlethwaite type theorem connecting the BR polynomial to the arrow polynomial.

References

Dye, H. (2017). Cut points: An invariant of virtual links. Journal of Knot Theory and Its Ramifications, 26, 1743006.