The Conway-Gordon Theorems

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Outline

What is a Spatial Embedding?

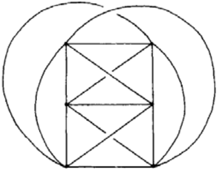
For our purposes, considering a graph as a topological object,

Definition

A **Spatial Embedding** of a graph G is the image of a injective continuous map $f : G \to \mathbb{R}^3$

Note that the Spatial Graph's vertices can take any shape we want when it is projected down to $\mathbb{R}^2$

A Spatial Embedding of K_6 projected to $\mathbb{R}^2$:



The Theorems in Question

Theorem

Every Spatial Embedding of K_6 contains a non-trivial link

Theorem

Every Spatial Embedding of K_7 contains a non-trivial knot

Idea of the Proof

Theorem

Every Spatial Embedding of K_6 contains a non-trivial link

The idea of this proof stems from this, which we will not prove:

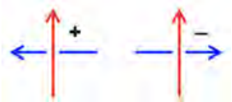
Lemma

Let G' and G'' be spatial embeddings of the same graph. Then, G' can be transformed to G'' by a series of crossing changes and isotopies

This means that if we can find an invariant that doesn't change over both these operations, we can get information about all possible spatial embeddings from a single embedding!

Linking Number

Each crossing in a projection of an oriented link can be classified as positive or negative by rotating the the crossing to match one of the following:



The *linking number* of a 2-component link is calculated by assigning each positive crossing a value of $+1/2$ and each negative crossing a value of $-1/2$, and then adding these values over all crossings of the two components with eachother.

Linking Number

Since changing the orientation of a component will switch the sign of every crossing, doing so will multiply the linking number by -1. Therefore orientation changes only affect the sign of the linking number.

Because of this, the linking number of an unoriented link is defined as the absolute value of the linking number obtained by assigning arbitrary orientations to each component.

Linking Number

For the proof of theorem 1, we'll be considering the linking number mod 2, so let $\mathrm{lk}(L_1, L_2)$ denote the linking number of L_1 and L_2 mod 2.

The Proof, Part 1

The invariant we will consider is

$$\Omega = \sum_{(L_1, L_2)} \mathrm{lk}(L_1, L_2) \pmod 2$$

Where L_1 and L_2 are two disjoint triangles in a Spatial Embedding of K_6. This invariant clearly doesn't change over isotopy, as the linking number doesn't change over isotopies.

Part 2

Now let's consider the effect that a crossing change has on Ω. There are some cases to consider:

1. The crossing is between an edge with itself
2. The crossing is between two adjacent edges
3. The crossing is between two non-adjacent distinct edges

Since the calculation of linking number only considers crossings involving both components, crossings of the first two types do not contribute to the linking number, thus changing those crossings does not affect Ω. (Adjacent edges cannot be part of different triangles since L_1 and L_2 are disjoint.)

Part 2, Third Case

Now, we must consider the third case: the crossing is between two non-adjacent distinct edges. This can only affect Ω when one edge is in L_1 and the other is in L_2.

Since a crossing change switches the sign of that crossing, the contribution of this crossing to the linking number will switch from $\mp 1/2$ to $\pm 1/2$. Thus any pair (L_1, L_2) which has this crossing between L_1 and L_2 will have its linking number change by ± 1 due to the crossing change.

Part 2, Third Case

Next, for any non-adjacent distinct edges A, B, we must count how many pairs of disjoint triangles (L_1, L_2) in K_6 there are such that A is in one triangle and B is in the other.

Without loss of generality, assume that $A \subset L_1$ and $B \subset L_2$.

Label the vertices of the graph $v_1, \ldots, v_6$ such that A connects v_1 and v_2, and B connects v_3 and v_4. It can be seen that there are two pairs (L_1, L_2) which meet these conditions:

$$\{v_1, v_2, v_5\} \subset L_1, \{v_3, v_4, v_6\} \subset L_2$$

$$\{v_1, v_2, v_6\} \subset L_1, \{v_3, v_4, v_5\} \subset L_2$$

Part 2, Third Case

Thus for any pair of non-adjacent distinct edges A, B, performing a crossing change results in all affected pairs (L_1, L_2) changing their crossing number by ± 1, and there are two pairs (L_1, L_2) affected by this change, so Ω (the sum of linking numbers of all pairs mod 2) is changed by a multiple of 2 due to the crossing change.

Since $\Omega \in \mathbb{Z}_2$, this means Ω is unchanged by a crossing change between two non-adjacent distinct edges.

Therefore Ω is also invariant under crossing changes.

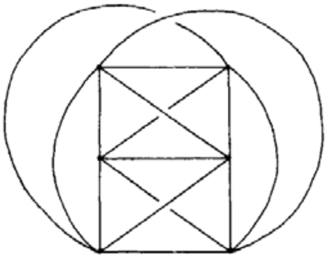
Part 3

Since any spatial embedding of a graph can be transformed to any other embedding of the same graph using only crossing changes and isotopies (from the Lemma at beginning of proof), it follows that Ω is the same for all possible spatial embeddings of K_6.

If we can show that some spatial embedding of K_6 has $\Omega = 1$, then every spatial embedding of K_6 must have $\Omega = 1$, which means that every spatial embedding contains at least 1 pair of distinct triangles (L_1, L_2) for which $\text{lk}(L_1, L_2) = 1$. Since an odd linking number must be nonzero, and the trivial link (unlink) has linking number zero, this means that L_1 and L_2 are non-trivially linked.

Part 3

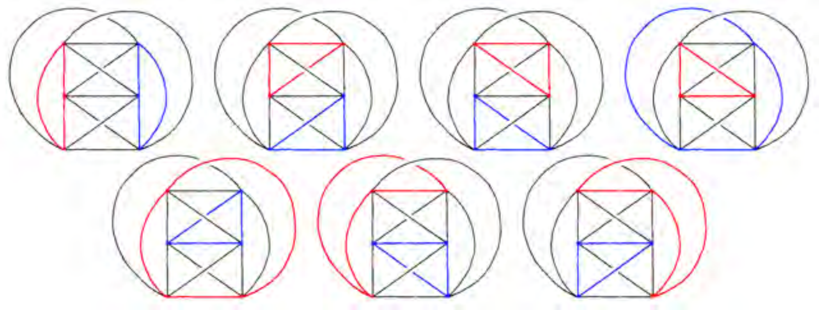
Consider the following spatial embedding of K_6 projected down to $\mathbb{R}^2$.



To calculate the value of Ω for this embedding, we must find the linking number of all $\frac{1}{2}\binom{6}{3} = 10$ pairs of disjoint triangles.

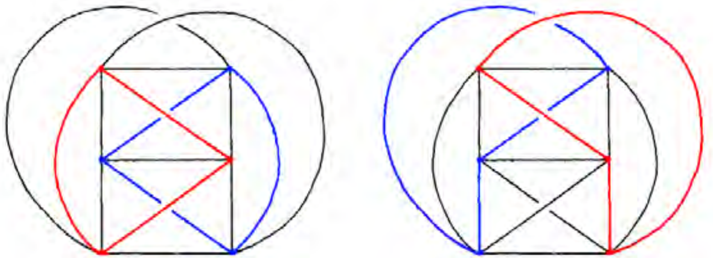
Part 3

These seven pairs of triangles do not cross at all, thus each pair is the trivial link (linking number zero).



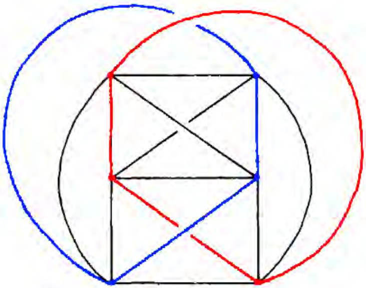
Part 3

These two pairs do cross, but in both cases the red triangle is clearly on top of the other, so both pairs are the trivial link again.



Part 3

The final pair of triangles is linked with linking number 1.



Hence $\Omega = 1$ for this embedding, and thus all embeddings of K_6.

This completes the proof of Theorem 1.

Idea of the Proof

Theorem

Every Spatial Embedding of K_7 contains a non-trivial knot

The idea is much the same as in proof 1:

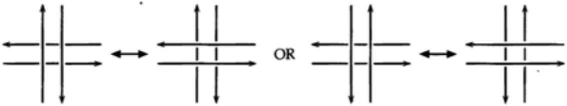
1. Find an invariant over Isotopy and Crossing changes
2. Find a Spatial Embedding of K_7 that has a good value for the invariant

The Invariant in Question

The critical element of the last proof was to find an invariant over Isotopy and Crossing changes. In that proof, we used the sum of linking numbers mod 2.

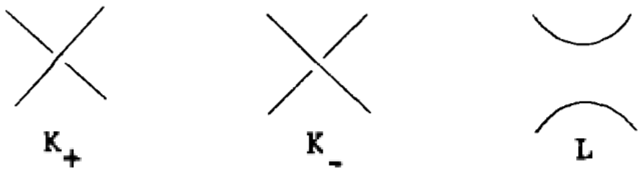
In this proof, we're going to be using the sum of the *arf* invariant of knots, denoted $\alpha(\gamma)$ for a knot γ.

In general, the definition can be seen in a couple ways, and is relatively complicated, so we'll only mention the important facts about it.



The Arf Invariant's Special Property

If K is a knot, for any given crossing we can define $K_+ = K$ and K_- given by switching the crossing. Finally, we can produce a link by splitting the crossing as shown:



Let L_1, L_2 be the components of L. Then,

$$\alpha(K_+) = \alpha(K_-) + \mathrm{lk}(L_1, L_2) \pmod 2$$

The Spatial Embedding Invariant

As you can guess, the invariant we will use for this proof is

$$S = \sum_{\gamma} \alpha(\gamma) \pmod 2,$$

where the sum is over all Hamiltonian cycles γ in a spatial embedding of K_7.

Just as before it is clear that S is invariant over isotopies, as the arf invariant is invariant over isotopies.

Three Cases

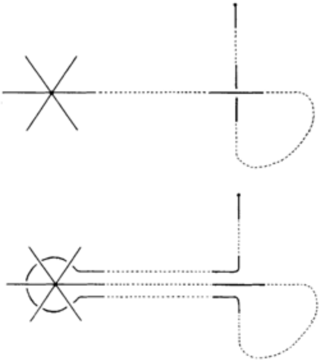
Like in the proof for Theorem 1, we have three cases for crossing changes:

1. The crossing is between an edge with itself
2. The crossing is between two adjacent edges
3. The crossing is between two non-adjacent distinct edges

This time, we can't ignore case 1 as easily, and we can't ignore case 2 at all.

Ignoring Case 1

If an edge has a self crossing, we can perform the following move to change it into a bunch of crossings between distinct edges:



Further Remarks on the Linking Number

Let $\omega(L_1, L_2)$ be the number of times L_1 crosses over L_2 mod 2. It can be shown that for any link, $\omega(L_1, L_2) = \mathrm{lk}(L_1, L_2) \pmod 2$.

In this theorem we are considering links which are formed out of edges of a graph. So, for two edges A, B, let $\omega(A, B)$ be the number of times A crosses over B mod 2. Then,

$$\mathrm{lk}(L_1, L_2) = \sum_{A \subset L_1, B \subset L_2} \omega(A, B) \pmod 2$$

What are We counting?

For any crossing change, if γ is a Hamiltonian cycle that contains both edges involved in that crossing, then

$$\alpha(\gamma) = \alpha(\gamma') + \mathrm{lk}(L_1, L_2) \pmod 2$$

If γ does not contain both edges involved, then $\alpha(\gamma)$ is unchanged. So,

$$S = S' + \sum_{\gamma} \mathrm{lk}(L_1, L_2) \pmod 2$$

where the sum is over all Hamiltonian cycles γ which contain both edges involved in the crossing.
Ultimately, we're going to be counting the number of such γ in each case.

Counting Arguments, Part 1

Suppose A, B are distinct adjacent edges. Then, through isotopy, we can locally get the crossing between them to look like this:

In this case, L_1 consists of one edge and L_2 is the rest of the γ not involved in the crossing change, so

$$\operatorname{lk}(L_1, L_2) = \sum_{\substack{E \subset \gamma \\ E \neq A,B}} \omega(L_1, E) \pmod 2$$

$$S = S' + \sum_{\gamma \supset A,B} \sum_{\substack{E \subset \gamma \\ E \neq A,B}} \omega(L_1, E) \pmod 2$$

Part 1 Cont.

$$S = S' + \sum_{\gamma \supset A,B} \sum_{\substack{E \subset \gamma \\ E \neq A,B}} \omega(L_1, E) \pmod 2$$

We can switch the order of summation, becoming

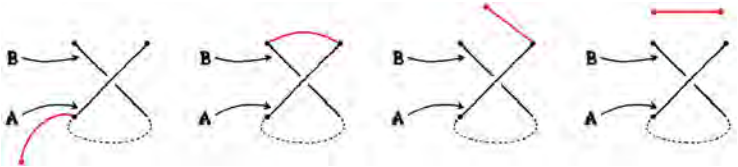
$$S = S' + \sum_{E \neq A,B} \left[\sum_{\gamma \supset E,A,B} \omega(L_1, E) \right] \pmod 2$$

$$S = S' + \sum_{E \neq A,B} \left[\omega(L_1, E) \sum_{\gamma \supset E,A,B} 1 \right] \pmod 2$$

So, to show $S = S'$, it is sufficient for there to be an even number of $\gamma \supset E, A, B$.

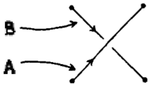
Part 1 Cont.

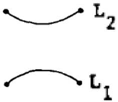
We will resolve all possible cases. First, if E, A, B have a common vertex, then trivially, the number of Hamiltonian cycles is 0 (as the same vertex would have to be revisited to traverse all 3 of them).
Similarly, if E is adjacent to both A and B, there are no Hamiltonian cycles.
Suppose that E is adjacent to exclusively one of A, B. Without loss of generality, we consider the A case.
Otherwise,
Thus, as there are an even number of γ, it follows that $S = S'$.



Part 2

Now, suppose we have two distinct edges. Then, the crossing and resulting link look like





As in part 1,

$$\operatorname{lk}(L_1, L_2) = \sum_{\substack{E_1 \subset L_1 \\ E_2 \subset L_2}} \omega(E_1, E_2) \pmod 2,$$

so

$$S = S' + \sum_{\gamma \supset A,B} \sum_{\substack{E_1 \subset L_1 \\ E_2 \subset L_2}} \omega(E_1, E_2) \pmod 2$$

Part 2

Again, we can switch summations to get

$$S = S' + \sum_{E_1, E_2 \neq A, B} \left[\sum_{\gamma \supset E_1, E_2, A, B} \omega(E_1, E_2) \right] \pmod 2$$

$$S = S' + \sum_{E_1, E_2 \neq A, B} \left[\omega(E_1, E_2) \sum_{\gamma \supset E_1, E_2, A, B} 1 \right] \pmod 2$$

Here, the summation is over unordered pairs $\{E_1, E_2\}$. So it suffices to show that the number of paths $\gamma \supset E_1, E_2, A, B$ is even for all possible E_1, E_2.

Part 2 Cont.

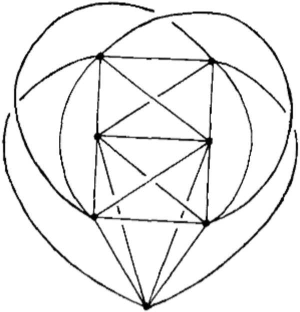
End of Proof

From the previous slides, we have shown that the sum of the Arf invariants S is invariant under isotopy and crossing changes, thus by the same Lemma used in the first theorem, S is the same for all spatial embeddings for K_7.

Following the process of the first theorem, all that is left is to verify that $S = 1$ for some embedding of K_7

End of Proof

Consider the following spatial embedding of K_7 projected to $\mathbb{R}_2$.



It will be shown that this embedding has $S = 1$.

Calculating S

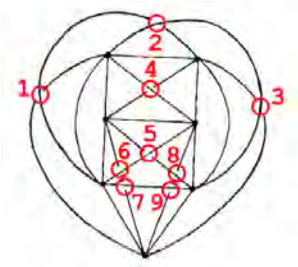
One way to determine S would be to list all $\frac{1}{2}6! = 360$ Hamiltonian cycles, then determine what knot each one forms.

However, in this case almost all cycles create the unknot, which has an arf invariant of zero, thus does not contribute to S.

Therefore it will be easier to find all nontrivial knots which are also Hamiltonian cycles using another method (which does not involve checking hundreds of cycles).

Calculating S

First, label each crossing:



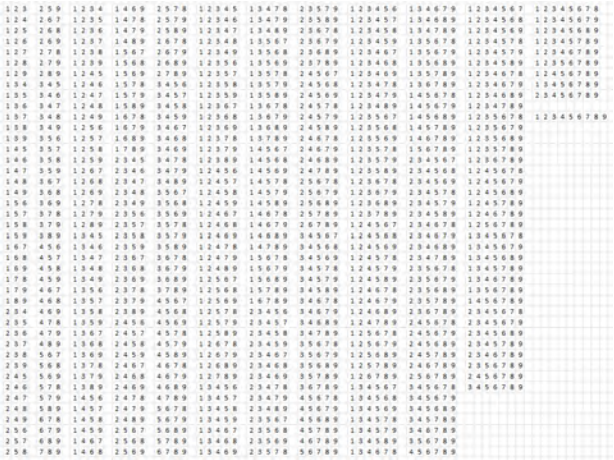
Every Hamiltonian cycle uses some (possibly empty) subset of these 9 crossings.

Calculating S

Every nontrivial knot has a crossing number of at least 3.

Since we want to find all nontrivial knots, we must consider every subset of those 9 crossings that contains at least 3 elements.
A complete list of all such subsets is:

Calculating S



123	259	1234	1469	2578	12345	13478	23579	123456	134679	1234567	12345678
124	267	1235	1478	2579	12346	13479	23589	123457	134689	1234568	12345679
125	268	1236	1479	2589	12347	13489	23678	123458	134789	1234569	12345689
126	269	1237	1489	2678	12348	13567	23679	123459	135678	1234578	12345789
127	278	1238	1567	2679	12349	13568	23689	123467	135679	1234579	12346789
128	279	1239	1568	2689	12356	13569	23789	123468	135689	1234589	12356789
129	289	1245	1569	2789	12357	13578	24567	123469	135789	1234678	12456789
134	345	1246	1578	3456	12358	13579	24568	123478	136789	1234679	13456789
135	346	1247	1579	3457	12359	13589	24569	123479	145678	1234689	23456789
136	347	1248	1589	3458	12367	13678	24578	123489	145679	1234789	
137	348	1249	1678	3459	12368	13679	24579	123567	145689	1235678	123456789
138	349	1256	1679	3467	12369	13689	24589	123568	145789	1235679	
139	356	1257	1689	3468	12378	13789	24678	123569	146789	1235689	
145	357	1258	1789	3469	12379	14567	24679	123578	156789	1235789	
146	358	1259	2345	3478	12389	14568	24689	123579	234567	1236789	
147	359	1267	2346	3479	12456	14569	24789	123589	234568	1245678	
148	367	1268	2347	3489	12457	14578	25678	123678	234569	1245679	
149	368	1269	2348	3567	12458	14579	25679	123679	234578	1245689	
156	369	1278	2349	3568	12459	14589	25689	123689	234579	1245789	
157	378	1279	2356	3569	12467	14678	25789	123789	234589	1246789	
158	379	1289	2357	3578	12468	14679	26789	124567	234678	1256789	
159	389	1345	2358	3579	12469	14689	34567	124568	234679	1345678	
167	456	1346	2359	3589	12478	14789	34568	124569	234689	1345679	
168	457	1347	2367	3678	12479	15678	34569	124578	234789	1345689	
169	458	1348	2368	3679	12489	15679	34578	124579	235678	1345789	
178	459	1349	2369	3689	12567	15689	34579	124589	235679	1346789	
179	467	1356	2378	3789	12568	15789	34589	124678	235689	1356789	
189	468	1357	2379	4567	12569	16789	34678	124679	235789	1456789	
234	469	1358	2389	4568	12578	23456	34679	124689	236789	2345678	
235	478	1359	2456	4569	12579	23457	34689	124789	245678	2345679	
236	479	1367	2457	4578	12589	23458	34789	125678	245679	2345689	
237	489	1368	2458	4579	12678	23459	35678	125679	245689	2345789	
238	567	1369	2459	4589	12679	23467	35679	125689	245789	2346789	
239	568	1378	2467	4678	12689	23468	35689	125789	246789	2356789	
245	569	1379	2468	4679	12789	23469	35789	126789	256789	2456789	
246	578	1389	2469	4689	13456	23478	36789	134567	345678	3456789	
247	579	1456	2478	4789	13457	23479	45678	134568	345679		
248	589	1457	2479	5678	13458	23489	45679	134569	345689		
249	678	1458	2489	5679	13459	23567	45689	134578	345789		
256	679	1459	2567	5689	13467	23568	45789	134579	346789		
257	689	1467	2568	5789	13468	23569	46789	134589	356789		
258	789	1468	2569	6789	13469	23578	56789	134678	456789		

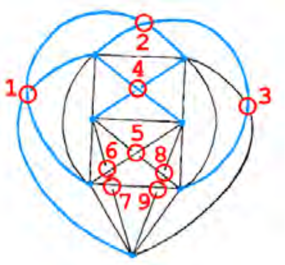
Calculating S

Checking all of these for a valid Hamiltonian cycle would be unreasonable (in fact, there are more subsets here than Hamiltonian cycles).

However, nearly all of these subsets can be eliminated without individually checking them, since most are incompatible a Hamiltonian cycle.

Calculating S

Consider a cycle which uses crossings 1, 2, and 4 (and possibly others). The following edges are necessary for those crossings:



This combination of crossings requires 3 edges that meet at the top left vertex, so no cycles use crossings 1, 2, and 4 together.

Calculating S

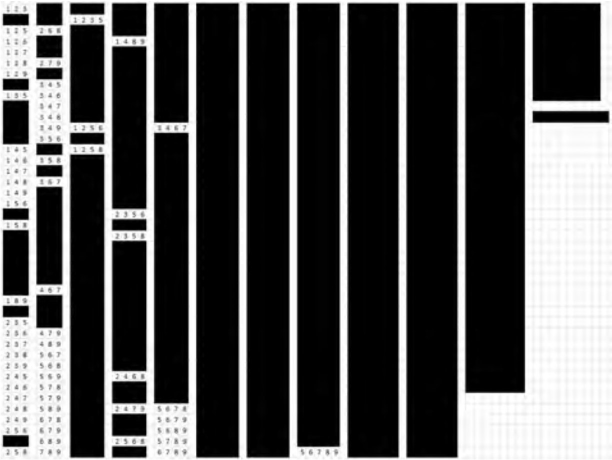
This means that any subset of the crossings that contains 1, 2, and 4 can be rejected, as no Hamiltonian cycle has those crossings.

The following triples all result in 3 edges meeting at a single vertex:

1 2 4	1 6 7	2 6 9	3 8 9
1 3 4	1 6 8	2 7 8	4 5 6
1 3 6	1 6 9	2 8 9	4 5 7
1 3 7	1 7 8	3 5 7	4 5 8
1 3 8	1 7 9	3 5 9	4 5 9
1 3 9	2 3 4	3 6 8	4 6 8
1 5 7	2 5 7	3 6 9	4 6 9
1 5 9	2 5 9	3 7 8	4 7 8
	2 6 7	3 7 9	

Thus any subset of the crossings containing any of these triples can be rejected:

Calculating S



Calculating S

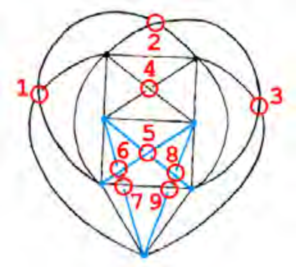
These are the only remaining crossing subsets.

1 2 3	1 5 6	2 5 6	4 7 9	1 2 3 5	2 5 6 8
1 2 5	1 5 8	2 5 8	4 8 9	1 2 5 6	3 4 6 7
1 2 6	1 8 9	2 6 8	5 6 7	1 2 5 8	5 6 7 8
1 2 7	2 3 5	2 7 9	5 6 8	1 4 8 9	5 6 7 9
1 2 8	2 3 6	3 4 5	5 6 9	2 3 5 6	5 6 8 9
1 2 9	2 3 7	3 4 6	5 7 8	2 3 5 8	5 7 8 9
1 3 5	2 3 8	3 4 7	5 7 9	2 4 6 8	6 7 8 9
1 4 5	2 3 9	3 4 8	5 8 9	2 4 7 9	
1 4 6	2 4 5	3 4 9	6 7 8		
1 4 7	2 4 6	3 5 6	6 7 9		
1 4 8	2 4 7	3 5 8	6 8 9		
1 4 9	2 4 8	3 6 7	7 8 9		
	2 4 9	4 6 7			5 6 7 8 9

This is still too many to check, so we must find more ways to remove large groups.

Calculating S

Consider a cycle which uses crossings 6 and 8. The following edges are necessary:



It can be seen that if crossings 6 and 8 are used, then 5 must also be used. Thus any crossing subset that contains 6 and 8 but not 5 can be rejected.

Calculating S

Similarly, all of the following are also invalid:

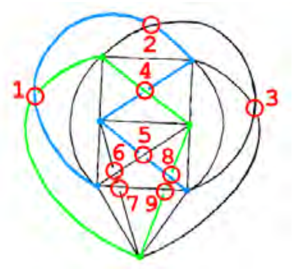
5,7 without 6 6,9 without 7 5,9 without 8 7,8 without 9

Using this information, the valid subsets are reduced to:

1 2 3	1 5 6	2 5 6	4 7 9	1 2 3 5	2 5 6 8
1 2 5	1 5 8	2 5 8	4 8 9	1 2 5 6	3 4 6 7
1 2 6	1 8 9	[redacted]	5 6 7	1 2 5 8	[redacted]
1 2 7	2 3 5	2 7 9	5 6 8	1 4 8 9	
1 2 8	2 3 6	3 4 5	[redacted]	2 3 5 6	
1 2 9	2 3 7	3 4 6		2 3 5 8	
1 3 5	2 3 8	3 4 7		[redacted]	
1 4 5	2 3 9	3 4 8	5 8 9	2 4 7 9	
1 4 6	2 4 5	3 4 9	[redacted]		
1 4 7	2 4 6	3 5 6	6 7 9		
1 4 8	2 4 7	3 5 8	[redacted]		
1 4 9	2 4 8	3 6 7	7 8 9		
	2 4 9	4 6 7			5 6 7 8 9

Calculating S

Next, consider a cycle which uses crossings 1,4, and 8. The following edges are necessary:



It can be seen that the green edges form a closed loop that does not pass through all vertices. Thus there cannot be a Hamiltonian cycle using crossings 1, 4, and 8.

Calculating S

All of the following subsets form loops that don't contain all vertices, thus cannot be used in a Hamiltonian cycle:

$$148, 149, 346, 347, 1256, 1489, 2358, 3467, 56789$$

Removing these, the valid subsets are reduced to:

1 2 3	1 5 6	2 5 6	4 7 9	1 2 3 5	2 5 6 8
1 2 5	1 5 8	2 5 8	4 8 9	[redacted]	[redacted]
1 2 6	1 8 9	[redacted]	5 6 7	1 2 5 8	
1 2 7	2 3 5	2 7 9	5 6 8	[redacted]	
1 2 8	2 3 6	3 4 5	[redacted]	2 3 5 6	
1 2 9	2 3 7	[redacted]		[redacted]	
1 3 5	2 3 8				
1 4 5	2 3 9	3 4 8	5 8 9	2 4 7 9	
1 4 6	2 4 5	3 4 9	[redacted]		
1 4 7	2 4 6	3 5 6	6 7 9		
[redacted]	2 4 7	3 5 8	[redacted]		
	2 4 8	3 6 7	7 8 9		
	2 4 9	4 6 7			[redacted]

Calculating S

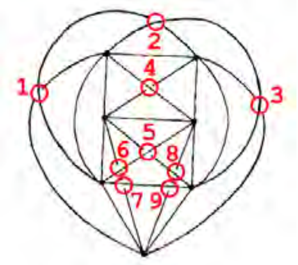
Most of the remaining subsets only have 3 crossings. The only nontrivial knot which can be drawn with 3 crossings is the trefoil knot.

Whenever the trefoil knot is drawn with only 3 crossings, it must be alternating. This means that as you travel around the knot, you never have 2 overcrossings or 2 undercrossings in a row.

In other words, any knot with has only 3 crossings and is not alternating must be the unknot.

Calculating S

For example, if a 3-crossing cycle contains crossings 2 and 4, it will have 2 overcrossings in a row, so will be the unknot.

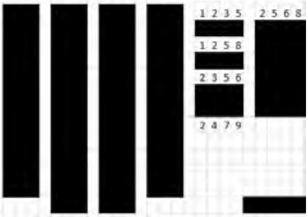


1 3	2 3	3 9
1 5	2 4	4 5
1 6	2 8	5 6
1 7	2 9	6 7
1 8	3 4	7 9
1 9	3 8	8 9

All of the above pairs result in a non-alternating knot, thus if they appear in a 3-crossing cycle, the resulting knot is trivial and can be removed.

Calculating S

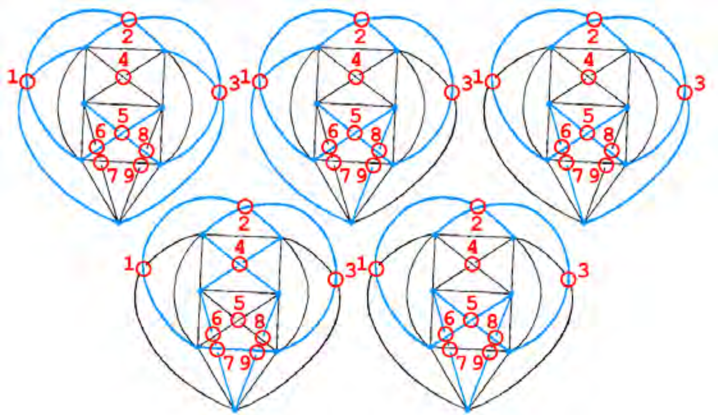
Using this information, we can remove most of the remaining subsets:



At this point there are only 5 sets of crossings left, so they can easily be examined individually.

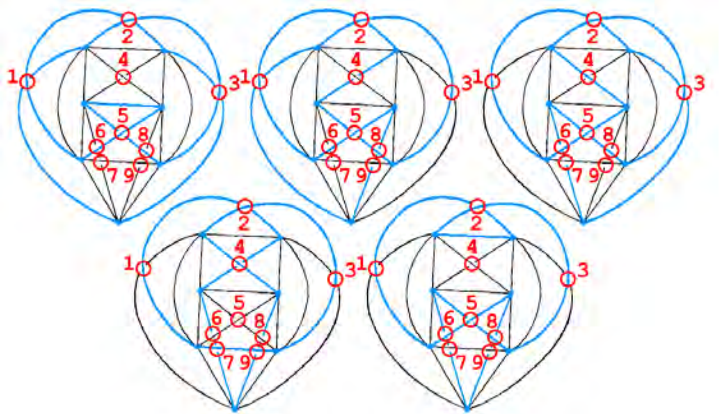
Calculating S

Filling in the necessary edges for each crossing set:



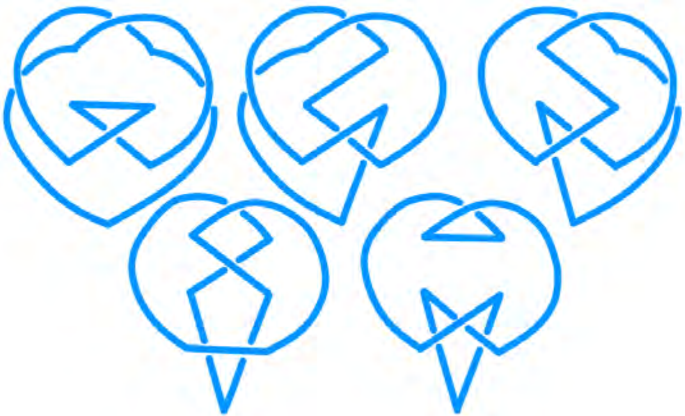
Calculating S

Each has 6 or 7 edges already filled, so there is only 1 way to complete each to a Hamiltonian cycle:



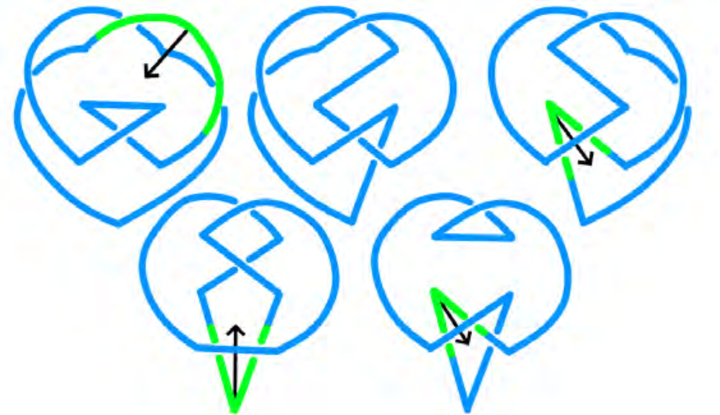
Calculating S

Removing unnecessary edges and widening the edges:



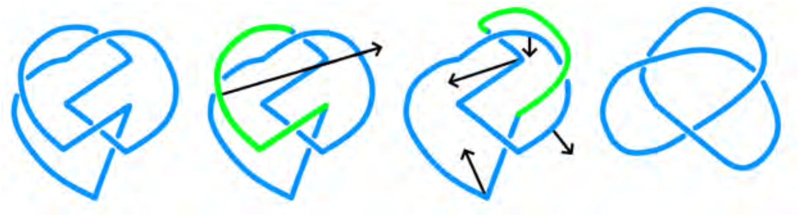
Calculating S

Four of the cycles can easily be unknotted:



Calculating S

The final cycle is a trefoil knot:



Thus out of all possible Hamiltonian cycles, one is a trefoil knot and the rest are the unknot. Since the unknot has an arf invariant of 0 and the trefoil knot has an arf invariant of 1, this means that the sum of arf invariants $S = 1$, as we claimed.

This completes the proof of Theorem 2.

Partial Theorem 2 for K_n

Slightly upgrading the counting argument given in theorem 2, it can be shown that

Theorem

The sum of the Arf invariants over all Hamiltonian Cycles of K_n is the same for all Spatial Embeddings of K_n for all $n \geq 7$.

The case of $n = 7$ was shown in the proof of theorem 2.