A Twisted Thistlethwaite Theorem

Joseph Paugh, Justin Wu, Gavin Zhang

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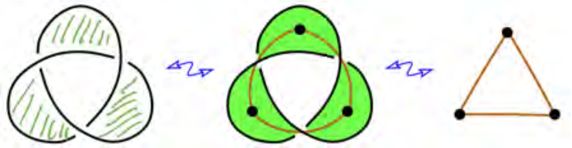
The Classic Thistlethwaite Theorem

- *Thistlethwaite's Theorem:* Up to sign and multiplication by a power of t, the Jones polynomial $J_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_\Gamma(-t, -t^{-1})$.



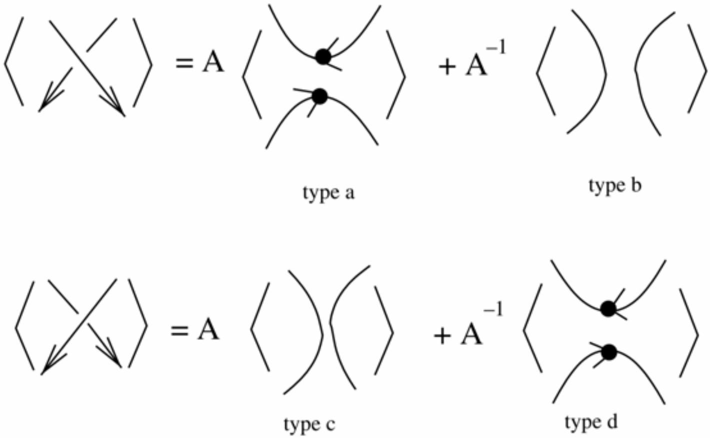
- In this presentation, Thistethwaite's theorem is extended to an abstract link diagram embedded in a potentially non-orientable surface, represented by a *Twisted Link Diagram*.

Introducing the Twisted Link

- Intuitively, twisted links are abstract links in oriented thickenings of potentially non-orientable surfaces.
- Twisted links may be represented by a *twisted link diagram*, consisting of an ordinary link diagram with circled crossings and bars.

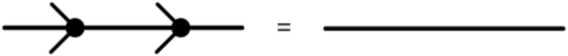
States of The Arrow Polynomial For Twisted Links

- For twisted link diagrams, we obtain a state using the following oriented state expansion:

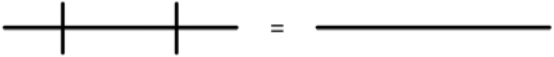


Reducing States

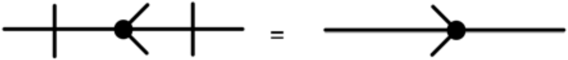
- After smoothing, the result is a collection of loops, potentially decorated with arrows and bars. We reduce these loops as follows:
- Two adjacent arrows pointing in the same direction "cancel."



- We also use the "T2 Twisted Reidemeister move:"



- Finally, we introduce an arrow reduction rule involving bars:



Reducing States

- If a loop C has an even number of bars, then $\langle C \rangle = K_n$, where $2n$ is the number of arrows in C after being reduced.
  - $K_0 = 1$ by convention.
- If a loop C has an odd number of bars, we set $\langle C \rangle = M$.
- Then $\langle S \rangle = \prod_{C \in S} \langle C \rangle$ for any state S.

Definition

The *arrow bracket polynomial* $\langle D \rangle_A$ of an oriented twisted link D is the polynomial in $\mathbb{Z}[A, B, d, M, K_1, K_2, \dots]$ defined by

$$\langle D \rangle_A(A, B, d) = \sum_S A^\alpha B^\beta d^{|S|-1} \langle S \rangle,$$

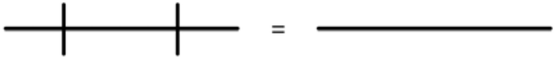
where:

- α is the number of A splittings in S;
- β is the number of B splittings in S;
- $|S|$ is the number of loops in S.

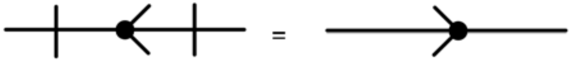
- The quantity $(-A^3)^{-w(D)} \langle D \rangle_A(A, A^{-1}, -A^2 - A^{-2})$ is an invariant of twisted links called the *normalized arrow polynomial* of D.

An Analogy Between Bars and Arrows

- Only two moves are involved in reducing bars in a loop:
  1. The T2 Twisted Reidemeister Move:

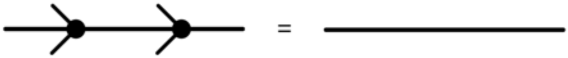


  2. The new bar reduction move:

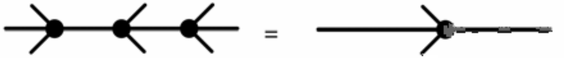


An Analogy Between Bars and Arrows

- We have the same rules if we replace bars with arrows pointing along the orientation of D:
  1. Two adjacent arrows (pointing in the same direction):



  2. Oppositely oriented arrows on either side of an arrow:



An Analogy Between Bars and Arrows

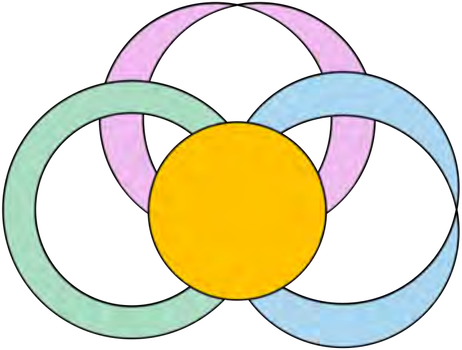
- Suppose we replace every bar with arrow nodes pointing along the orientation of D.
- For every loop C, we set $\langle C \rangle = K_{\frac{n}{2}}$ if C has n arrows after reduction.
  - Once again, $K_0 = 1$ by convention.
- Set $\langle S \rangle = \prod_{C \in S} \langle C \rangle$.
- Then $\sum_S A^\alpha B^\beta d^{|S|-1} \langle S \rangle$ still gives the arrow bracket polynomial, but with M replaced by $K_{1/2}$.

An Analogy Between Bars and Arrows

- This analogy between bars and arrows is key to our construction.
- Before using this observation, we first need to review ribbon graphs.

Ribbon Graphs

- Informally, a *ribbon graph* is a kind of topological graph, with vertices as discs and edges as ribbons:



- We will consider *arrow ribbon graphs*, whose vertices and edges may feature arrows along their topological boundary.

The Arrow Dichromatic Polynomial

Definition

For an arrow ribbon graph G, the arrow dichromatic polynomial A_G is given by

$$A_G(a, \boldsymbol{b}, c, \boldsymbol{K}) = \sum_{F \subseteq E(G)} a^{k(F)} \left(\prod_{e \in F} b_e \right) c^{bc(F)} \prod_{f \in \partial(F)} K_{i(f)}$$

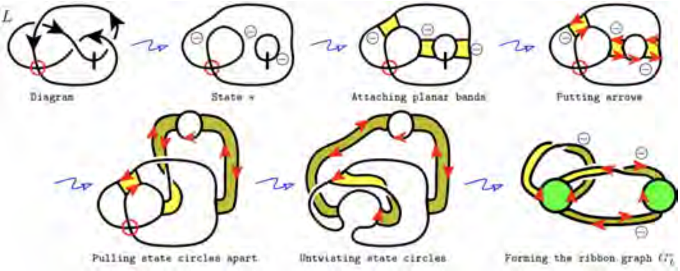
where:

- $\{b_e\}$ = "edge weights";
- $k(F) = \#$ of connected components of F;
- $bc(F) = \#$ of connected components of $\partial(F)$;
- $i(f)$ = half the reduced $\#$ of arrows on f.

Note that this sum is taken over spanning subgraphs F of G.

The Arrow Thistlethwaite Theorem

- For a state s of the diagram L, we form a signed ribbon graph G_L^s:



- Splittings → edges; State circles → vertices.
- A splitting → positive edge; B splitting → negative edge.
- Oriented smoothing → arrow pair along free edge arcs; Disoriented smoothing → arrow pair along attaching arcs.
- Bars → arrows along free edges of vertex discs.

The Twisted Thistlethwaite Theorem

Theorem

The arrow bracket polynomial of a twisted link diagram L is a specialization of the arrow dichromatic polynomial of G_L^s:

$$\langle L\rangle_A(A,B,d) = \frac{A^{e_+}B^{e_-}}{d} A_{G_L^s}(1,\boldsymbol{b},d,\boldsymbol{K})$$

where:

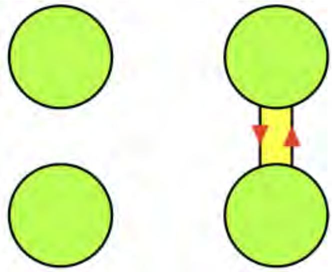
- $e_+ = \#$ *of positive edges in* G_L^s;
- $e_- = \#$ *of negative edges in* G_L^s;
- $b_e = \begin{cases} B/A & \text{if } e \text{ is positive} \\ A/B & \text{if } e \text{ is negative} \end{cases}.$

Proof Idea

- There is a one-to-one correspondence between spanning subgraphs F of G_L^s and states s' of L:
  - $s' \leftrightarrow$ spanning subgraph F that contains only the edges corresponding to the crossings of L where s' differs from s.
- Claim: the boundary components of F are the state circles of s', and hence carry the same arrow structure.
- To see why this is true, consider the following example.

Proof Idea

- Suppose we have two state circles connected by a ribbon, corresponding to an oriented smoothing.



- If we remove the edge (left), the boundary follows the splitting of our initial state, and features no arrows.
- If we keep the edge (right), the boundary follows the opposite disoriented splitting of our initial state, complete with arrows.

Proof Idea

- Thus the monominals contributed by s' and F to their respective polynomials have the same K_i terms.
- The other factors also work out:
  - The exponent of d is $bc(F) - 1 = |s'| - 1$.
  - The exponent of A is $e_+ - e_+(F) + e_-(F) = e_+[E(G_L^s)\backslash F] + e_-(F) = \alpha(s')$.
  - The exponent of B is $e_- - e_-(F) + e_+(F) = e_-[E(G_L^s)\backslash F] + e_+(F) = \beta(s')$.

- Depending on the initial choice of state, we can obtain different types of Thistlethwaite theorems for various polynomials.
- For example, choosing the Seifert state of the link diagram can be used to produce an analogous theorem for the arrow version of Riordan-Bollobas polynomial.

Bibliography

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