Thompson's Group

Project 2

June 28, 2021

Outline

F

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- The topic of this presentation is the so-called "Thompson's Group."
- There are in fact three different groups this may refer to: in the literature, the group we focus on today is referred to as "F", so we shall use "F" to refer to this group.

What is F?

Definition

The Thompson Group F is the set of piecewise linear homeomorphisms from $[0, 1]$ to itself, which are differentiable at every point except at a finite set of dyadic rationals, and such that the derivative is a power of 2 wherever it exists.

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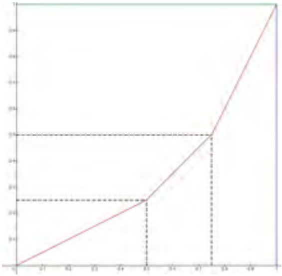
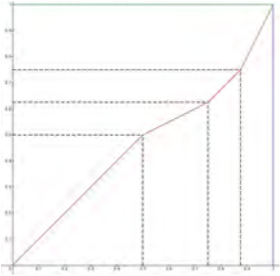
- A "dyadic rational" is simply a rational number $\frac{a}{b}$ (assumed to be fully reduced) such that b is a power of 2. Examples include $\frac{3}{4}$, $\frac{17}{256}$, and $\frac{1}{2}$.



The element $A(x) =$

$$\begin{cases} \frac{x}{2} & 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{4} & \frac{1}{2} \leq x \leq \frac{3}{4} \\ 2x - 1 & \frac{3}{4} \leq x \leq 1 \end{cases}$$



The element $B(x) =$

$$\begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4} & \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{8} & \frac{3}{4} \leq x \leq \frac{7}{8} \\ 2x - 1 & \frac{7}{8} \leq x \leq 1 \end{cases}$$

An important fact

- An important fact is that the two previous examples, $A(x)$ and $B(x)$, generate F.

Theorem

$F \cong \langle A, B \mid [AB^{-1}, A^{-1}BA] = [AB^{-1}, A^{-2}BA^2] = 1 \rangle$

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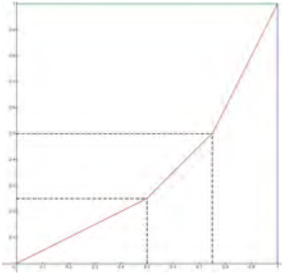

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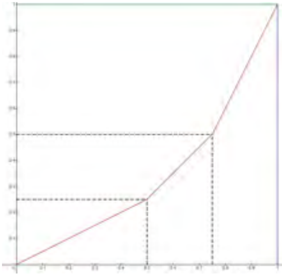

- We note that we can partition the unit interval into intervals on which $A(x)$ is linear: $[0, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{4}]$, and $[\frac{3}{4}, 1]$.

An Observation (continued)

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- Essentially, all A does is scale dyadic intervals (meaning intervals whose endpoints are both dyadic rationals) to other dyadic intervals.
- This is indeed true of every element of F, so we can represent each element as two sets of intervals: one for the domain and one for the range.

Binary Trees

- We will now discuss a way to represent every element of F as a pair of rooted binary trees

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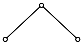
Definition

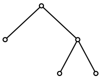
A rooted binary tree is a tree with one root, v_0, with valence 2, such that every non-root node has valence either 1 (in which case it is a leaf) or 3.

- Here are some examples

Examples



The simplest rooted binary tree, called a "caret"



A rooted binary tree with 3 leaves

Partitioning [0, 1] using rooted binary trees

- We can use a binary tree to represent a particular way to partition $[0, 1]$ into dyadic intervals.

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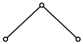
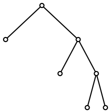
- We can use a binary tree to represent a particular way to partition $[0, 1]$ into dyadic intervals.
- Each caret can be interpreted as follows: the root of the caret is a particular dyadic interval $[a, b]$, with the leaves of the caret being $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$. A tree consisting of one caret would be the partition of the unit interval into $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$.

Examples



This is a partition of $[0,1]$ into $[0,\frac{1}{2}]\cup[\frac{1}{2},1]$



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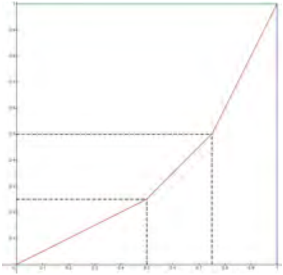

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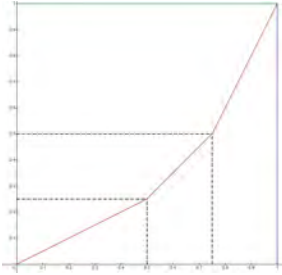
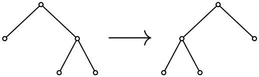


$A(x)$

- Looking at the dotted lines, we can see the domain is partitioned into $[0, \frac{1}{2}] \cup [\frac{1}{2}, \frac{3}{4}] \cup [\frac{3}{4}, 1]$, and the range is partitioned into $[0, \frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}] \cup [\frac{1}{2}, 1]$.

Representations

- Here is a representation of $A(x)$ as a pair of rooted binary trees:



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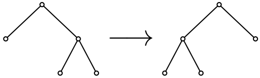
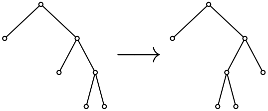


- And here's $B(x)$



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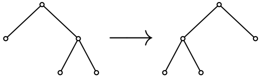

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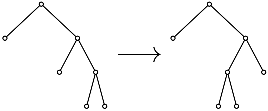

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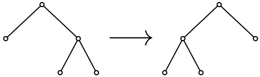

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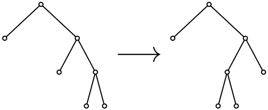

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- The reason for this is because we can think of elements of F acting on the set of rooted binary trees: if f sends the tree X to the tree Y, then in some sense $f \cdot X = Y$, so, by abusing notation, we can convince ourselves that $f = \frac{Y}{X}$ *in some sense*.

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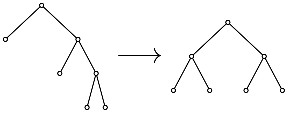

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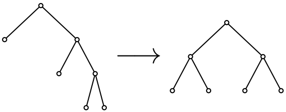
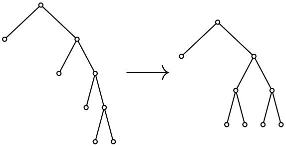


- And here' another for $B(x)$:



Representations, continued

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- For any element of F the two binary trees will have the same number of leaves
- It may not be immediately obvious that any such pair of trees represents an element of F.

Theorem

Every pair of rooted binary trees with the same number of leaves represents a valid element of F

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- $L_A = \{[0, a_1], [a_1, a_2], \ldots, [a_{n-1}, a_n]\}$ and $L_B = \{[0, b_1], [b_1, b_2], \ldots, [b_{n-1}, b_n]\}$, where when $i < j$ we have $a_i < a_j$ and $b_i < b_j$, and the a_i and b_i are all positive.

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- If $F \ni f = A \longrightarrow B$, then $f([a_i, a_{i+1}]) = [b_i, b_{i+1}]$ for $1 \leq i \leq n-1$, $f(0) = 0$ and $f(1) = 1$.

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- It is simple to construct such an f (or at least, to construct the graph) as follows: let the point $L_j = (a_j, b_j)$ for $1 \leq j \leq n-1$, $L_0 = (0, 0)$, and $L_n = (1, 1)$. If we draw a line from L_j to L_{j+1} for all $1 \leq j \leq n-1$, this graph will define an element of Thompson's group.

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- The points of nondifferentiability occur at dyadic rationals: recall that the a_i, b_i are dyadic by construction.

Proof of Theorem (Continued)

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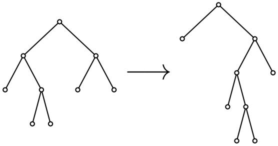
- Let's work through a simple example

Example

- 

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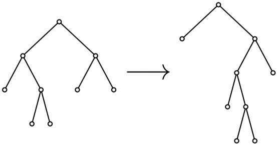
- 
- The first tree is a partition of the unit interval into $[0, \frac{1}{4}] \cup [\frac{1}{4}, \frac{3}{8}] \cup [\frac{3}{8}, \frac{1}{2}] \cup [\frac{1}{2}, \frac{3}{4}] \cup [\frac{3}{4}, 1]$, and the second is a partition of the unit interval into $[0, \frac{1}{2}] \cup [\frac{1}{2}, \frac{5}{8}] \cup [\frac{5}{8}, \frac{11}{16}] \cup [\frac{11}{16}, \frac{3}{4}] \cup [\frac{3}{4}, 1]$

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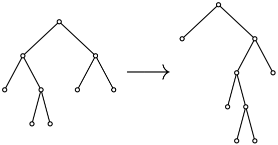
- 
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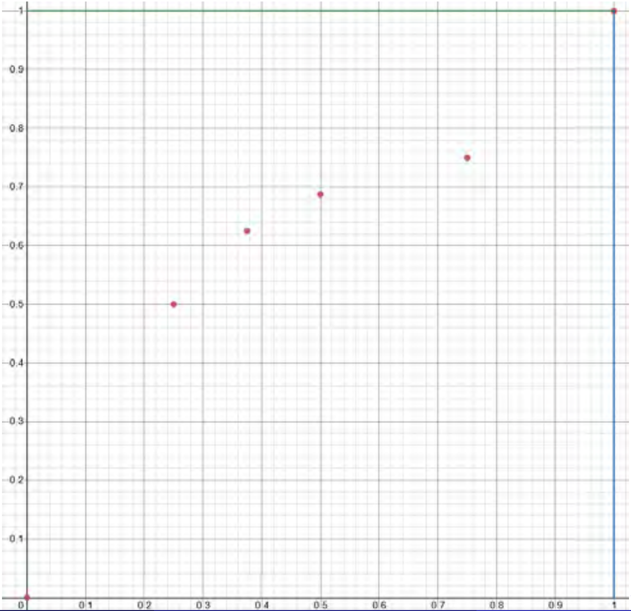
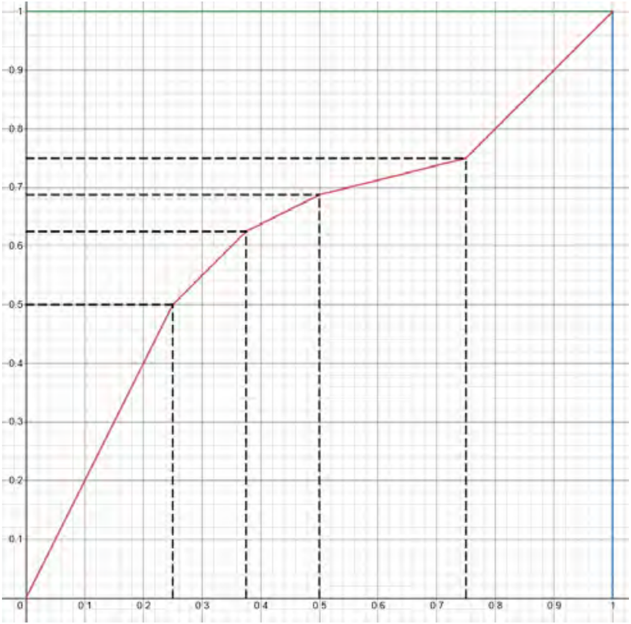
- So the points of nondifferentiability are $(\frac{1}{4}, \frac{1}{2})$, $(\frac{3}{8}, \frac{5}{8}), (\frac{1}{2}, \frac{11}{16}), (\frac{3}{4}, \frac{3}{4})$.

Example





From left to right, the function has slope 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, 1

Frame Title

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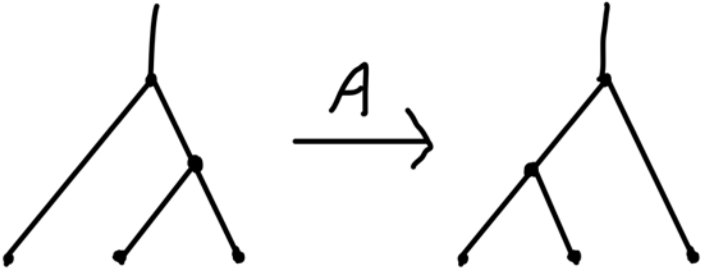
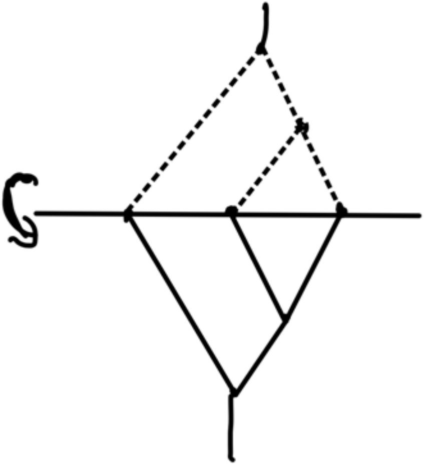
- There is a way to associate any element of F with a link. We will now discuss this.
- The best way to describe this algorithm is to just show how it's done, so let's start by doing it to $A(x)$

We Draw A(x) In a New Way



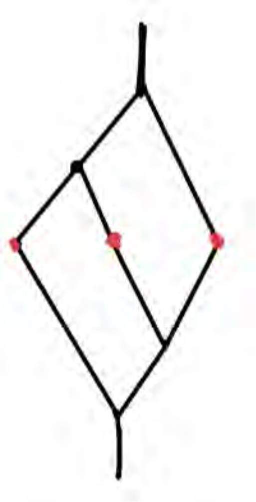
Here we have drawn $A(x)$ like this to make the next step easier to visualize

Flip the Denominator



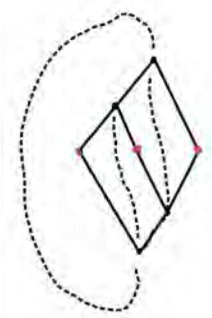
We reflect the denominator about the x-axis. Reminder: the denominator is the first tree

Put Them Together



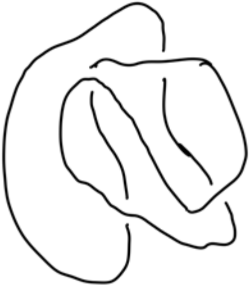
Next we connect the two trees like so. The red dots are the leaves.

Connecting Lines



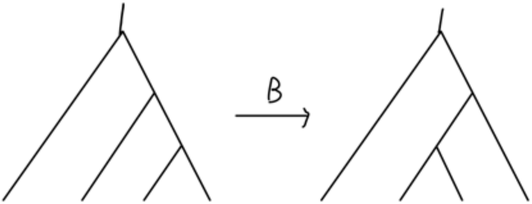
We draw lines like so. Dotted lines are always undercrossings

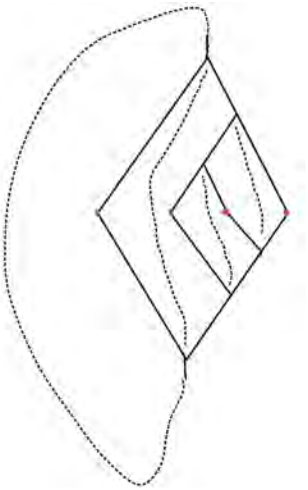
$\longrightarrow$



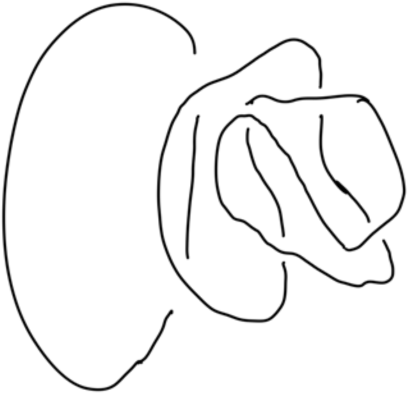
This process results in a link diagram of the unknot. We say it has Thompson index 3

Let's do B(x)



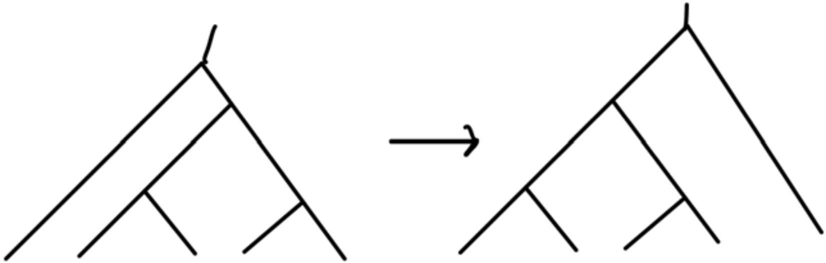


The Result

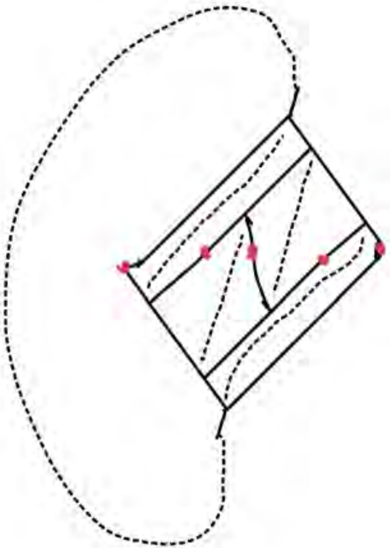


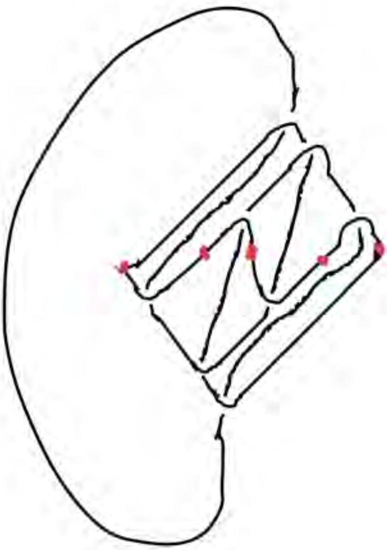
Note that this is just the diagram from A with an extra unknot which is not linked in any way

One More



Note this element of F has 5 leaves





The Result

The Trefoil!

The Obvious Question

- This begs an obvious question:

The Obvious Question

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Obvious Question

Is every link the result of this process for some element of F?

- Yes!

The Obvious Question

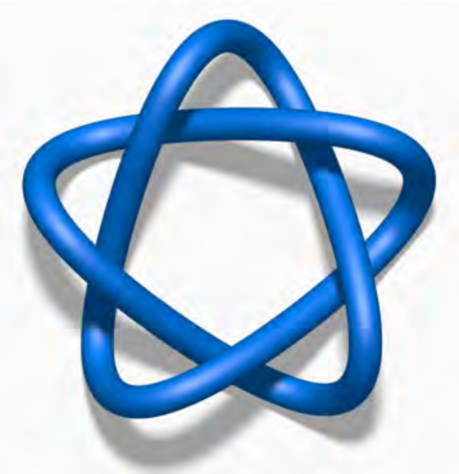
- This begs an obvious question:

Obvious Question

Is every link the result of this process for some element of F?

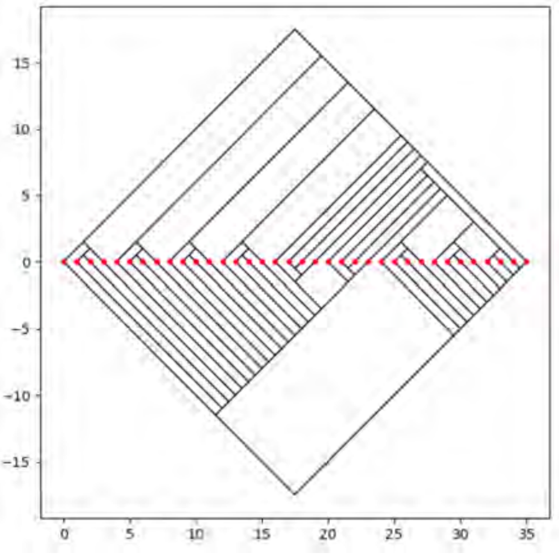
- Yes!
- Vaugh Jones came up with an algorithm to generate an element of F for any link. However, it is inefficient.

The 5_1 knot



This is the so-called "cinquefoil knot." Any guesses as to how many leaves the algorithm produces?

Lotta leaves



21 leaves

Conclusion

- This leaves us with the following link invariant:

Definition

The **Thompson index** of a link is minimal number of leaves required for an element of F to be associated to that link.

Conclusion

- This leaves us with the following link invariant:

Definition

The **Thompson index** of a link is minimal number of leaves required for an element of F to be associated to that link.

- One possible direction for future research is to find some kind of bound for the Thompson index in terms of the crossing number