The Conjugacy Problem in F

Project 2

July 23, 2021

The Conjugacy Problem

Definition

Two elements x, y of a group G are called *conjugate* if there is some $z \in G$ such that $x = z^{-1}yz$. Being conjugate is an equivalence relation, and the equivalence classes under this are called conjugacy classes.

- Very often in group theory, we will want to know if two elements are conjugate, or what the conjugacy classes of a particular group look like.

The Conjugacy Problem

Definition

Two elements x, y of a group G are called *conjugate* if there is some $z \in G$ such that $x = z^{-1}yz$. Being conjugate is an equivalence relation, and the equivalence classes under this are called conjugacy classes.

- Very often in group theory, we will want to know if two elements are conjugate, or what the conjugacy classes of a particular group look like.
- Answering these questions is known as the conjugacy problem.

The Conjugacy Problem

Definition

Two elements x, y of a group G are called *conjugate* if there is some $z \in G$ such that $x = z^{-1}yz$. Being conjugate is an equivalence relation, and the equivalence classes under this are called conjugacy classes.

- Very often in group theory, we will want to know if two elements are conjugate, or what the conjugacy classes of a particular group look like.
- Answering these questions is known as the conjugacy problem.
- In this talk, we will be using objects called strand diagrams to solve the conjugacy problem in the Thompson Group F.

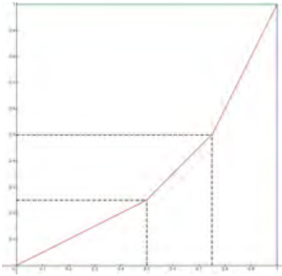
Thompson's Group

In case you've forgotten, here is a refresher:

Definition

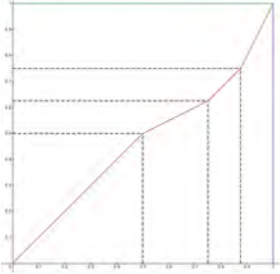
The Thompson Group is the group of piecewise-linear self-homeomorphisms of the $[0, 1]$ interval, which fix 0 and 1, such that it is differentiable except at a finite set of dyadic rationals, and such that the derivative (where it exists) is an integer power of 2. The group operation is function composition.

Examples



The element $A(x) =$

$$\begin{cases} \frac{x}{2} & 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{4} & \frac{1}{2} \leq x \leq \frac{3}{4} \\ 2x - 1 & \frac{3}{4} \leq x \leq 1 \end{cases}$$



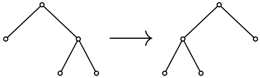
The element $B(x) =$

$$\begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4} & \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{8} & \frac{3}{4} \leq x \leq \frac{7}{8} \\ 2x - 1 & \frac{7}{8} \leq x \leq 1 \end{cases}$$

Examples of trees

We saw in a previous presentation that it is possible to associate with each element of F a pair of rooted binary trees, the denominator and the numerator.

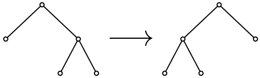
- Here is a representation of $A(x)$ as a pair of rooted binary trees:



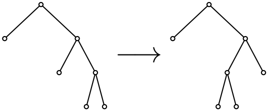
Examples of trees

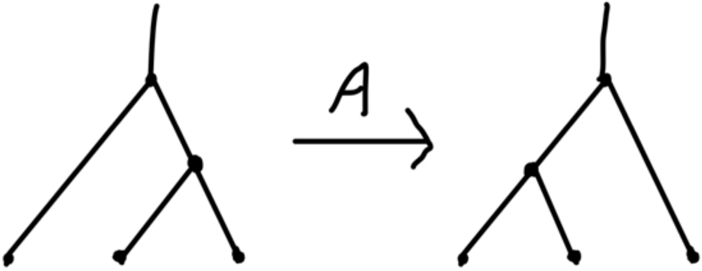
We saw in a previous presentation that it is possible to associate with each element of F a pair of rooted binary trees, the denominator and the numerator.

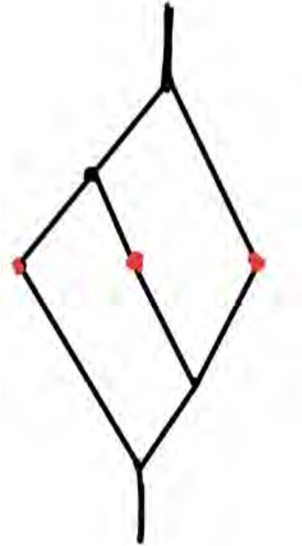
- Here is a representation of $A(x)$ as a pair of rooted binary trees:



- And here's $B(x)$




A



Tree Diagrams Aren't Perfect

- One problem with these is that, with respect to the group operation, these are difficult to interpret geometrically.

Tree Diagrams Aren't Perfect

- One problem with these is that, with respect to the group operation, these are difficult to interpret geometrically.
- That is, given two of these diagrams, it is a pain to figure out what the composition of the functions they represent is, outside of some nice cases.

Tree Diagrams Aren't Perfect

- One problem with these is that, with respect to the group operation, these are difficult to interpret geometrically.
- That is, given two of these diagrams, it is a pain to figure out what the composition of the functions they represent is, outside of some nice cases.
- To rectify this, there is another way of representing elements of F using what are called strand diagrams.

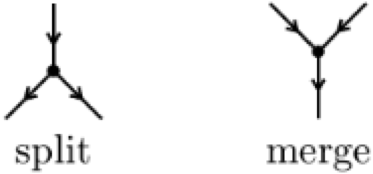
Strand Diagrams

Definition

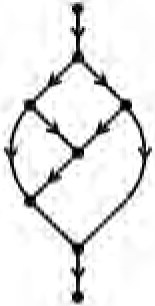
A *strand diagram* is any directed, acyclic graph in the unit square satisfying the following conditions:

- There exists a unique univalent source along the top of the square,and a unique univalent sink along the bottom of the square.
- Every other vertex lies in the interior of the square, and is either a split or a merge

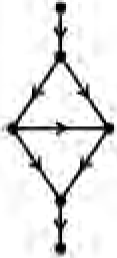
Split and Merge



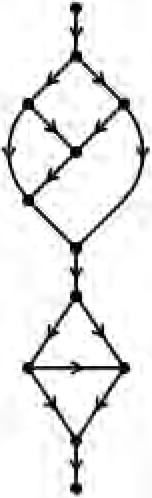
Examples



f



g



$g \circ f$

Reductions

Strand diagrams are subject to “moves” (similar to Reidemeister moves) called *reductions*.

A strand diagram is called “reduced” if no reductions can be performed

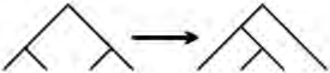
Theorem

Every strand diagram can be reduced to a unique reduced strand diagram.

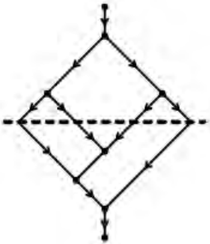
- Because of this theorem (which will be proven in the appendix at the end), the operation of composing two reduced strand diagrams and then reducing the result is well defined.

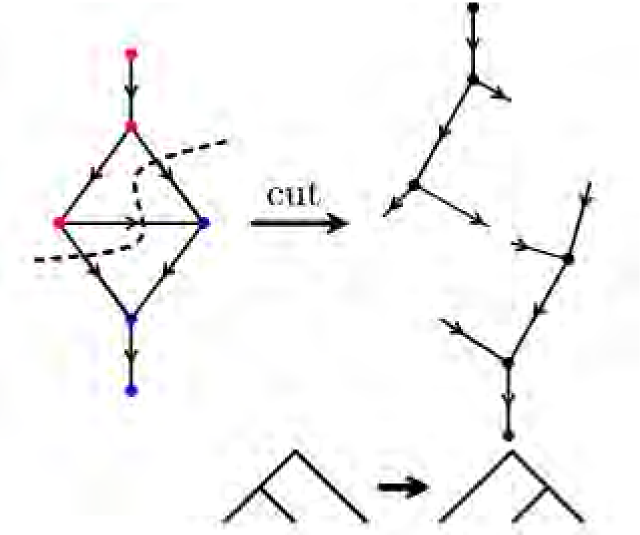
- Because of this theorem (which will be proven in the appendix at the end), the operation of composing two reduced strand diagrams and then reducing the result is well defined.
- So, under this operation, reduced strand diagrams form a group, with an elements inverse being obtained by flipping it about the x-axis and reversing the orientation.

- Because of this theorem (which will be proven in the appendix at the end), the operation of composing two reduced strand diagrams and then reducing the result is well defined.
- So, under this operation, reduced strand diagrams form a group, with an elements inverse being obtained by flipping it about the x-axis and reversing the orientation.
- It is also easy to go between strand diagrams and tree diagrams for any given element of F



becomes




cut

Very Good Theorem

Theorem

The group of reduced strand diagrams, with concatenation as the operation, is isomorphic to F

Proof of Very Good Theorem

- Let $\mathcal{SD}$ refer to the group of strand diagrams, let $\mathcal{TD}$ refer to the group of tree diagrams, and as usual let F refer to the thompson group.

Proof of Very Good Theorem

- Let $\mathcal{SD}$ refer to the group of strand diagrams, let $\mathcal{TD}$ refer to the group of tree diagrams, and as usual let F refer to the thompson group.
- We begin by constructing a homomorphism from $\mathcal{SD}$ to F. If D is a reduced strand diagram, we can use it to build an element of F, which we call f_D, like so.

Proof of Very Good Theorem

- Let $\mathcal{SD}$ refer to the group of strand diagrams, let $\mathcal{TD}$ refer to the group of tree diagrams, and as usual let F refer to the thompson group.
- We begin by constructing a homomorphism from $\mathcal{SD}$ to F. If D is a reduced strand diagram, we can use it to build an element of F, which we call f_D, like so.
- Given $t \in [0, 1]$, we calculate $f_D(t)$ as follows. First, we write t's binary expansion, $0.b_1 b_2 \ldots$.

Proof of Very Good Theorem

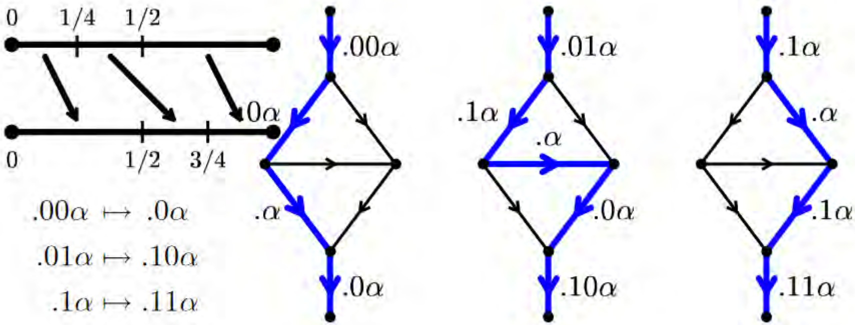
- Let $\mathcal{SD}$ refer to the group of strand diagrams, let $\mathcal{TD}$ refer to the group of tree diagrams, and as usual let F refer to the thompson group.
- We begin by constructing a homomorphism from $\mathcal{SD}$ to F. If D is a reduced strand diagram, we can use it to build an element of F, which we call f_D, like so.
- Given $t \in [0, 1]$, we calculate $f_D(t)$ as follows. First, we write t's binary expansion, $0.b_1 b_2 \ldots$.
- Then, we start at the source of D, following the edge. If we come to a split, we remove b_1, to obtain $0.b_2 b_3 \ldots$. If $b_1 = 0$, we move left out of the split, and if $b_1 = 1$, we move right.

Proof of Very Good Theorem

- Let $\mathcal{SD}$ refer to the group of strand diagrams, let $\mathcal{TD}$ refer to the group of tree diagrams, and as usual let F refer to the thompson group.
- We begin by constructing a homomorphism from $\mathcal{SD}$ to F. If D is a reduced strand diagram, we can use it to build an element of F, which we call f_D, like so.
- Given $t \in [0, 1]$, we calculate $f_D(t)$ as follows. First, we write t's binary expansion, $0.b_1 b_2 \ldots$.
- Then, we start at the source of D, following the edge. If we come to a split, we remove b_1, to obtain $0.b_2 b_3 \ldots$. If $b_1 = 0$, we move left out of the split, and if $b_1 = 1$, we move right.
- Now, if we have the number $0.a_1 a_2 \ldots$ and we meet a merge, we add a new digit to obtain $0.a_0 a_1 \ldots$. If we enter the merge from the left, $a_0 = 0$, and if we enter from the right, $a_0 = 1$.

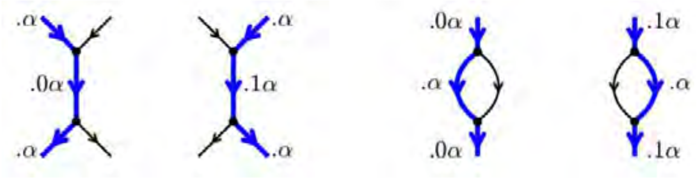
Proof of Very Good Theorem

- Let $\mathcal{SD}$ refer to the group of strand diagrams, let $\mathcal{TD}$ refer to the group of tree diagrams, and as usual let F refer to the thompson group.
- We begin by constructing a homomorphism from $\mathcal{SD}$ to F. If D is a reduced strand diagram, we can use it to build an element of F, which we call f_D, like so.
- Given $t \in [0,1]$, we calculate $f_D(t)$ as follows. First, we write t's binary expansion, $0.b_1 b_2 \dots$.
- Then, we start at the source of D, following the edge. If we come to a split, we remove b_1, to obtain $0.b_2 b_3 \dots$. If $b_1 = 0$, we move left out of the split, and if $b_1 = 1$, we move right.
- Now, if we have the number $0.a_1 a_2 \dots$ and we meet a merge, we add a new digit to obtain $0.a_0 a_1 \dots$. If we enter the merge from the left, $a_0 = 0$, and if we enter from the right, $a_0 = 1$.
- $f_D(t)$ is the result once we reach the end of the diagram.



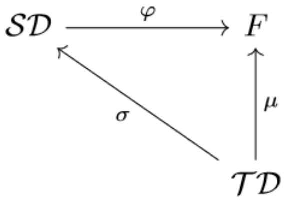
We can see that the map $\varphi : D \mapsto f_D$ is a homomorphism

Reductions Don't Change f_D



We can see here that reductions do not change the action of f_D on $[0, 1]$

Proof of Bijectivity



In this diagram, φ is the map described previously, σ is the map from tree diagrams to strand diagrams obtained by imposing an orientation on a tree diagram, and μ is the typical way we associate an element of F to a tree diagram.

Proof of Bijectivity

- First, we will show that there is only one way we can cut a strand diagram to obtain a tree diagram. Let $D \in \mathcal{SD}$, let v be a cut, and let T_v be the tree diagram obtained by this cut. I claim that T_v is in fact independent of v.

Proof of Bijectivity

- First, we will show that there is only one way we can cut a strand diagram to obtain a tree diagram. Let $D \in \mathcal{SD}$, let v be a cut, and let T_v be the tree diagram obtained by this cut. I claim that T_v is in fact independent of v.
- We have by definition that $\sigma(T_v) = D$. By the commutative diagram on the previous page, we have $\mu(T_v) = \varphi(\sigma(T_v)) = \varphi(D)$.

Proof of Bijectivity

- First, we will show that there is only one way we can cut a strand diagram to obtain a tree diagram. Let $D \in \mathcal{SD}$, let v be a cut, and let T_v be the tree diagram obtained by this cut. I claim that T_v is in fact independent of v.
- We have by definition that $\sigma(T_v) = D$. By the commutative diagram on the previous page, we have $\mu(T_v) = \varphi(\sigma(T_v)) = \varphi(D)$.
- We know μ is invertible, so we have $T_v = \mu^{-1}(\varphi(D))$, so the claim is proven.

Proof of Bijectivity

- First, we will show that there is only one way we can cut a strand diagram to obtain a tree diagram. Let $D \in \mathcal{SD}$, let v be a cut, and let T_v be the tree diagram obtained by this cut. I claim that T_v is in fact independent of v.
- We have by definition that $\sigma(T_v) = D$. By the commutative diagram on the previous page, we have $\mu(T_v) = \varphi(\sigma(T_v)) = \varphi(D)$.
- We know μ is invertible, so we have $T_v = \mu^{-1}(\varphi(D))$, so the claim is proven.
- Thus, the map $\psi : D \mapsto T_v$ is well-defined, and we have $\mu\psi = \varphi$.

Proof of Bijectivity

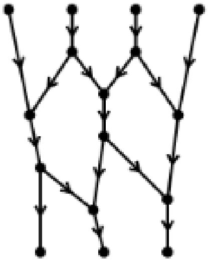
- First, we will show that there is only one way we can cut a strand diagram to obtain a tree diagram. Let $D \in \mathcal{SD}$, let v be a cut, and let T_v be the tree diagram obtained by this cut. I claim that T_v is in fact independent of v.
- We have by definition that $\sigma(T_v) = D$. By the commutative diagram on the previous page, we have $\mu(T_v) = \varphi(\sigma(T_v)) = \varphi(D)$.
- We know μ is invertible, so we have $T_v = \mu^{-1}(\varphi(D))$, so the claim is proven.
- Thus, the map $\psi : D \mapsto T_v$ is well-defined, and we have $\mu\psi = \varphi$.
- So, we can see that σ and ψ are inverses of each other, and so ψ is an isomorphism from $\mathcal{SD}$ to $\mathcal{TD}$, which is already isomorphic to F, so the proof is finished.

The Conjugacy Problem

Definition

An (m, n) strand diagram is similar to a strand diagram defined above, but with m sources and n sinks. In particular, every strand diagram discussed up to now is a $(1, 1)$ strand diagram.

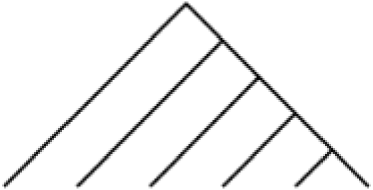
Here is an example:



A $(4, 3)$ strand diagram

(k, k) strand diagrams

Similarly to before, the set of (k, k) strand diagrams form a group, but these are isomorphic for any k.



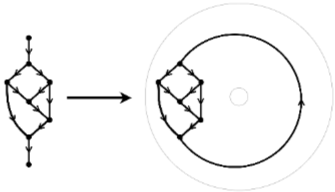
By gluing this to the top and bottom of a (k, k) strand diagram, we can easily see that the groups are in fact isomorphic.

Annular Strand Diagrams

Definition

An *annular strand diagram* is a directed graph embedded in an annulus with the following properties:

- Every vertex is either a merge or a split
- Every directed cycle has positive winding number around the central hole.



The definition allows for free loops unconnected to anything else, although this example doesn't have one

Annular Strand Diagrams

- Annular strand diagrams have what are called "cutting paths."

Annular Strand Diagrams

- Annular strand diagrams have what are called "cutting paths."
- They're exactly what they sound like: a cutting path is a path which goes from the center of the annulus to the edge, intersects none of the vertices, and intersects each edge transversely.

Annular Strand Diagrams

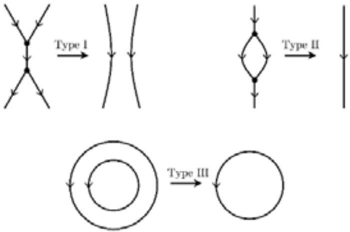
- Annular strand diagrams have what are called “cutting paths.”
- They’re exactly what they sound like: a cutting path is a path which goes from the center of the annulus to the edge, intersects none of the vertices, and intersects each edge transversely.
- A cutting path cuts an annular strand diagram into a (k, k) strand diagram, and so, by the previous note, an element of F.

Annular Strand Diagrams

- Annular strand diagrams have what are called "cutting paths."
- They're exactly what they sound like: a cutting path is a path which goes from the center of the annulus to the edge, intersects none of the vertices, and intersects each edge transversely.
- A cutting path cuts an annular strand diagram into a (k, k) strand diagram, and so, by the previous note, an element of F.
- Conversely, a (k, k) strand diagram can be associated to an annular strand diagram by gluing the ith sink to the ith source, similarly to the closure of a braid, if any of you remember that from last year.

Annular Reductions

Similarly as before, the annular strand diagrams are subject to "reductions."



Theorem

Every annular strand diagram reduces to a unique reduced annular strand diagram

The Solution To The Conjugacy Problem

At long last, here is the goal of this presentation:

Theorem

Two elements of F are conjugate if and only if they have the same reduced annular strand diagram.

- One direction is pretty easy

The Solution To The Conjugacy Problem

At long last, here is the goal of this presentation:

Theorem

Two elements of F are conjugate if and only if they have the same reduced annular strand diagram.

- One direction is pretty easy
- If two strand diagrams A, B are conjugate (that is there is some H such that $H^{-1}AH = B$), then we can use the fact that this has the same annular strand diagram as $AHH^{-1} = A$, so A and B have the same strand diagram.

The Solution To The Conjugacy Problem

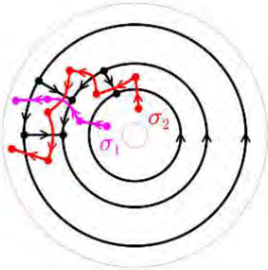
At long last, here is the goal of this presentation:

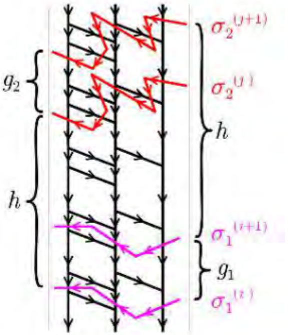
Theorem

Two elements of F are conjugate if and only if they have the same reduced annular strand diagram.

- One direction is pretty easy
- If two strand diagrams A, B are conjugate (that is there is some H such that $H^{-1}AH = B$), then we can use the fact that this has the same annular strand diagram as $AHH^{-1} = A$, so A and B have the same strand diagram.
- The other direction is trickier, and a proof of it will take up the rest of the presentation.

Distinct Cutting Paths Yield Conjugate Elements



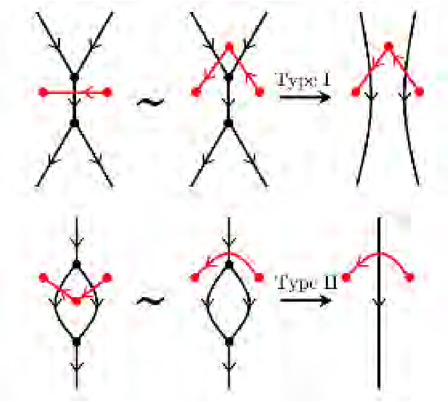


σ_1, σ_2 are cutting paths, and g_1, g_2 are the strand diagrams they induce. We can see $g_2 h = h g_1$, and so g_1, g_2 are conjugate

The upshot of this is that if two (k, k) strand diagrams determine the same annular strand diagram, then they are conjugate.

Reduction Doesn't Change Conjugacy Class

We can show that for any cutting path going through a reduction, it will determine the same strand diagram as cutting and then reducing.



This proves the theorem.

Diamond Lemma

Earlier in the talk, these two theorems were taken for granted:

Theorem

Every $((k, k))$ *strand diagram reduces to a unique reduced strand diagram*

Theorem

Every annular strand diagram reduces to a unique reduced annular strand diagram

This addendum will prove these theorems using objects from computer science called "abstract rewriting systems," and a property of these called "confluence".

Abstract Rewriting System

Definition

For our purposes, an *abstract rewriting system* is a directed graph, where objects represent states, and arrows represent rewriting.

- For example, if the expression a can be rewritten as b, then we would say $a \to b$.

Abstract Rewriting System

Definition

For our purposes, an *abstract rewriting system* is a directed graph, where objects represent states, and arrows represent rewriting.

- For example, if the expression a can be rewritten as b, then we would say $a \to b$.
- If there is a path $c \to c' \to c'' \to \cdots \to d$, we say $c \dot{\to} d$

Abstract Rewriting System

Definition

For our purposes, an *abstract rewriting system* is a directed graph, where objects represent states, and arrows represent rewriting.

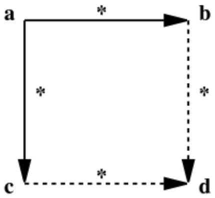
- For example, if the expression a can be rewritten as b, then we would say $a \to b$.
- If there is a path $c \to c' \to c'' \to \cdots \to d$, we say $c \dot{\to} d$
- Here is a basic example:
  $(11+9) \times (2+4) \to (20) \times (2+4) \to (20) \times (6)$, so
  $(11+9) \times (2+4) \dot{\to} (20) \times (6)$.

Confluence

Now, we may define confluence:

Definition

A node a in an abstract rewriting system is *confluent* if for all b, c such that $a \dot{\rightarrow} b$ and $a \dot{\rightarrow} c$, there is a d such that $b \dot{\rightarrow} d$ and $c \dot{\rightarrow} d$. If every a is confluent, then the abstract rewriting system is said to have the diamond property. If instead we weaken the condition to say that if $a \rightarrow b$ and $a \rightarrow c$ then there is a d such that $b \dot{\rightarrow} d$ and $c \dot{\rightarrow} d$, we say that it is *locally confluent*.



The Diamond

Diamond Lemma

With all this out of the way, we can formulate Newman's Lemma, also called the Diamond Lemma:

Theorem

If an abstract rewriting system is

- *Terminating, meaning there is no infinite sequence of reductions* $a_1 \to a_2 \to a_3 \to \cdots$
- *Locally confluent*

then it is confluent. In particular, because of the first condition, there is a unique a in the system such that for any b, $b \dot{\to} a$.

Proof of Theorems

- We will first prove that any strand diagram D reduces to a unique reduced strand diagram.

Proof of Theorems

- We will first prove that any strand diagram D reduces to a unique reduced strand diagram.
- In this case, the objects are the strand diagrams which can be obtained from D by reductions and the rewritings are the reductions.

Proof of Theorems

- We will first prove that any strand diagram D reduces to a unique reduced strand diagram.
- In this case, the objects are the strand diagrams which can be obtained from D by reductions and the rewritings are the reductions.
- Each reduction reduces the number of vertices, so it is clearly terminating. So, we must show it is locally confluent, and we are done.

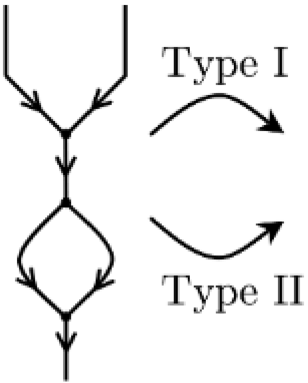
Proof of Theorems

- We will first prove that any strand diagram D reduces to a unique reduced strand diagram.
- In this case, the objects are the strand diagrams which can be obtained from D by reductions and the rewritings are the reductions.
- Each reduction reduces the number of vertices, so it is clearly terminating. So, we must show it is locally confluent, and we are done.
- Suppose two reductions have no vertices in common. In that case, they commute, and we have trivial local confluence.

Proof of Theorems

- We will first prove that any strand diagram D reduces to a unique reduced strand diagram.
- In this case, the objects are the strand diagrams which can be obtained from D by reductions and the rewritings are the reductions.
- Each reduction reduces the number of vertices, so it is clearly terminating. So, we must show it is locally confluent, and we are done.
- Suppose two reductions have no vertices in common. In that case, they commute, and we have trivial local confluence.
- The next figure illustrates what happens in the other case

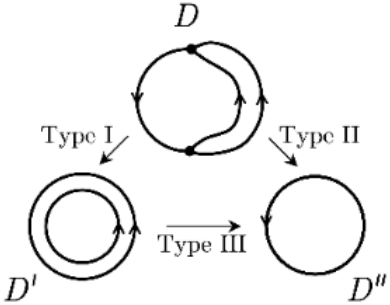
Local Confluence



We can see it is locally confluent, so we may conclude by Newman's Lemma that it is confluent, proving the assertion.

For Annular Strand Diagrams

In the case of annular strand diagrams, the situation is much the same, with one added case to worry about, which we show here



Thus, we have shown local confluence, so by Newman's Lemma we may conclue that it is confluent, proving the assertion.

Sources

- "Dynamics in Thompson's Group F" by Belk and Matucci https://core.ac.uk/download/pdf/13283033.pdf
- "ALGORITHMS AND CLASSIFICATION IN GROUPS OF PIECEWISE-LINEAR HOMEOMORPHISMS" by Matucci https://arxiv.org/pdf/0807.2871.pdf
- "CONJUGACY IN THOMPSON'S GROUPS" by Belk and Matucci https://core.ac.uk/download/pdf/13283032.pdf
- "CONJUGACY AND DYNAMICS IN THOMPSON'S GROUPS" by Belk and Matucci https://arxiv.org/pdf/0708.4250.pdf
- Wikipedia
- Internet al