The Conjugacy Problem In A Version of Thompson's Group

John White, in collaboration with Torey Hilbert, Alexander Patterson, Dennis Sweeny, Dr. Sergei Chmutov, and Dr. Matt Harper

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Groups (Optional)

A "group" is a mathematical object G consisting of a set, S, and a function $f : S^2 \to S$, such that the following axioms hold, with $f(a, b)$ denoted as $a \cdot b$, or just ab for short.

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- Identity: there exists an element called the "identity element," usually denoted e, such that for all $a \in G$, $e \cdot a = a \cdot e = a$
- Inverse: for any $a \in G$, there is an "inverse," usually denoted a^{-1}, such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

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- In a group, determining whether two elements are conjugate or not (that is, whether there is such a z as above), is called *the conjugacy problem*
- In this talk, I will be presenting a complete solution to the conjugacy problem in a certain group.

The Group In Question

Definition

We will denote by F_3 the group of piecewise linear homeomorphisms f of the $[0, 1]$ interval, which fix the endpoints, such that each f is differentiable except perhaps at a finite number of triadic rationals (rational numbers with a 3 in the denominator), and such that the derivative, where it exists, is always 3^k for some $k \in \mathbb{Z}$.

Checking

We quickly check that it is indeed a group:

- If f, g are two elements of F_3, then whenever $f \circ g$ is differentiable, it will locally be the composition of two linear functions with derivative a power of 3, so it will locally be such a function (alternatively, one could use the chain rule)

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- The points of non-differentiability of $f \circ g$ are a subset of the points of non-differentiability of f union the points of non-differentiability of g, so $f \circ g$ is only non-differentiable at a finite (possibly zero) set of ternary rationals.

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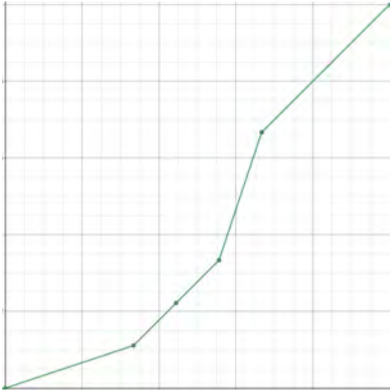
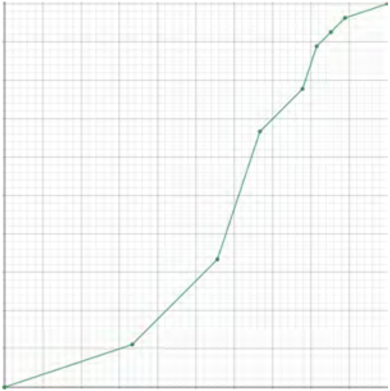
- The identity function, $f(x) = x$, satisfies this condition, as $1 = 3^0$
- Later we will show how to easily compute inverses, so we know it has inverses

Examples





Tree Diagrams

Each element of F_3 can be thought of in the following way:

- First we chop up the unit interval $[0, 1]$ into finitely many triadic intervals (intervals whose length is a power of 3) in two ways: $\{[a_0, a_1], [a_1, a_2], \ldots, [a_{n-1}, a_n]\}$ and $\{[b_0, b_1], [b_1, b_2], \ldots, [b_{n-1}, b_n]\}$, where $a_0 = b_0 = 0$ and $a_n = b_n = 1$, and such that each a_i and b_i is a power of 3.

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- Then, we have a function which linearly maps the ith interval in the first set to the ith interval in the second linearly. It will be differentiable except at the n points $(a_i, b_i)_{1 \leq i \leq n}$.

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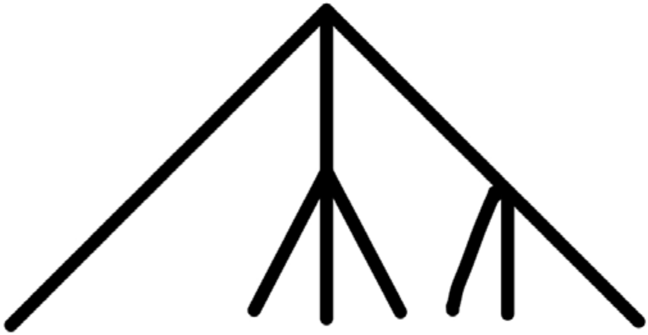
- This also lets us easily compute the inverse of a function by considering the function which maps the second set of intervals to the first.
- We can use what are called "rooted ternary trees" to represent ways of chopping up intervals

Rooted Ternary Trees

Definition

A rooted ternary tree is a tree with each vertex having valence either 1, 3, or 4, and with only one vertex, called the "root," having valence 3. Here is an example:



Using Tree Diagrams To Represent Elements of F_3

We can use a rooted ternary tree to represent a way of chopping the unit interval into ternary intervals, as discussed previously.

- The three edges coming off the root each represent an interval of length $\frac{1}{3}$. The first is $[0, \frac{1}{3}]$, the second is $[\frac{1}{3}, \frac{2}{3}]$, and the third is $[\frac{2}{3}, 1]$. Whenever one of those edges hits a splitting point, the interval that edge represents is split into three intervals, each of length $\frac{1}{3}$ the interval being split, as with the first split at the root.

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- So, every element of F_3 can be represented as a pair of rooted ternary trees, one for each set of intervals which define that element.

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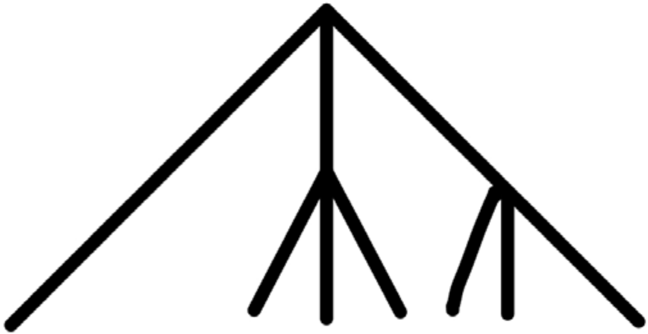
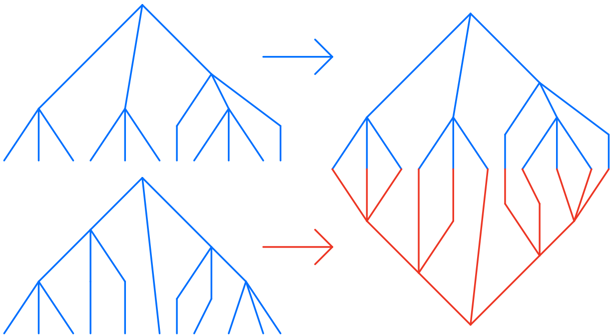
- So, every element of F_3 can be represented as a pair of rooted ternary trees, one for each set of intervals which define that element.
- Finally, given two such trees, we can unify them into a single diagram by flipping the second along the x-axis, and sticking it to the bottom of the other

Chopped Intervals



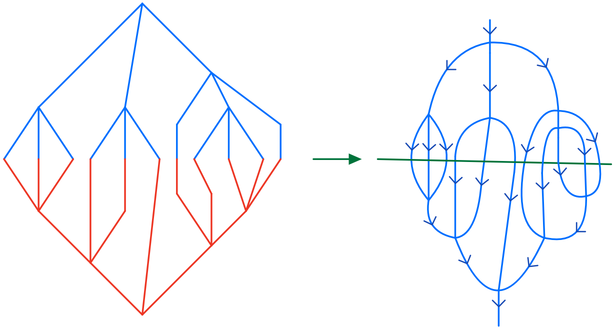
This represents a partition of $[0,1]$ into
$\{[0,\frac{1}{3}],[\frac{1}{3},\frac{4}{9}],[\frac{4}{9},\frac{5}{9}],[\frac{5}{9},\frac{2}{3}],[\frac{2}{3},\frac{7}{9}],[\frac{7}{9},\frac{8}{9}],[\frac{8}{9},1]\}$

Tree Diagram Example



Strand Diagrams

By imposing a downward orientation on these tree diagrams, we obtain what is called a "strand diagram."



Strand Diagrams

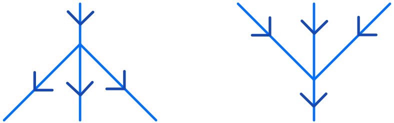
Definition

A "strand diagram" is a directed, acyclic graph in the unit square, satisfying the following two properties:

- There exists a univalent source along the top of the square, and a univalent sink along the bottom
- Every other vertex lies in the interior of the square, and is either a *split* or a *merge*

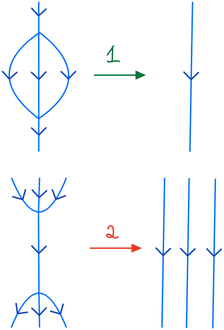
These were originally introduced by Belk and Matucci in 2008 to study the conjugacy problem in the classic Thompson's Group (which is defined very similarly to F_3), and we have adapted their work to F_3.

Splits & Merges

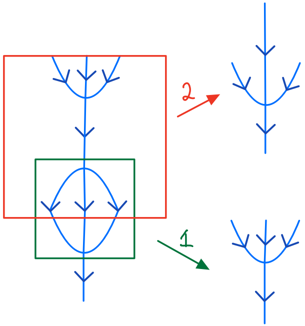


Reductions

Each strand diagram is subject to the following two "reductions"



Reduction 1 (top) and reduction 2 (bottom)

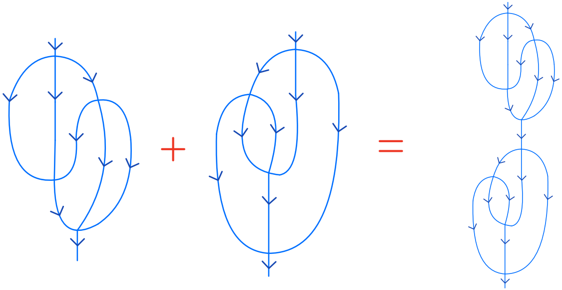


We can see that when the two reductions share a vertex, we can apply either reduction and we get the same thing

Because of this fact, we have that every strand diagram reduces to a unique "reduced" strand diagram (as in, for each strand diagram, we may talk about its "reduced form," and it is well defined)

Composition

We can "compose" two strand diagrams f, g into a single diagram, $f \circ g$



f, g, and $f \circ g$

Important Theorem

The following is a very good theorem

Theorem

The set of reduced strand diagrams forms a group, with the group operation sending (f, g) to the reduced form of $f \circ g$. Furthermore, this group is isomorphic to F_3!

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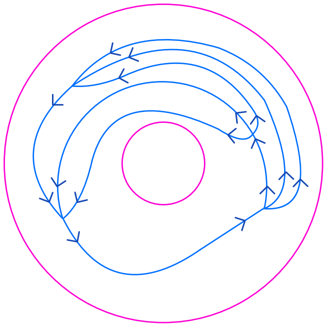
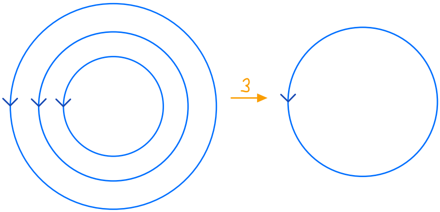
- We modify the third reduction found by Belk and Matucci and obtain a third reduction for these annular strand diagrams

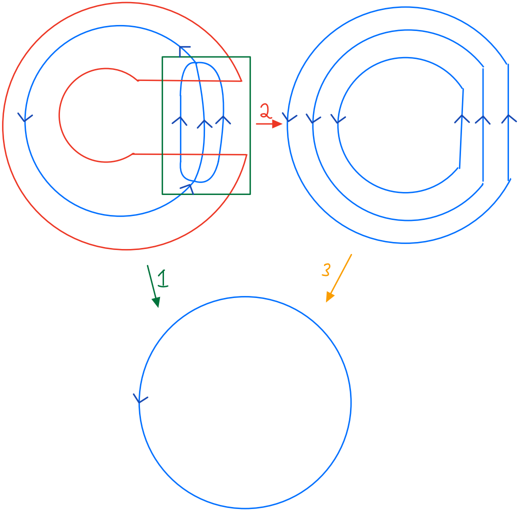
Annular Strand Diagram



The Third Reduction



The third reduction takes three concentric circles which do not interact and turns them into one circle.



2
1
3

The Result

Theorem

Two elements of F_3 are conjugate if and only if they correspond to the same reduced annular strand diagram.

The End

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Sources

- "Dynamics in Thompson's Group F" by Belk and Matucci https://core.ac.uk/download/pdf/13283033.pdf
- "ALGORITHMS AND CLASSIFICATION IN GROUPS OF PIECEWISE-LINEAR HOMEOMORPHISMS" by Matucci https://arxiv.org/pdf/0807.2871.pdf
- "CONJUGACY IN THOMPSON'S GROUPS" by Belk and Matucci https://core.ac.uk/download/pdf/13283032.pdf
- "CONJUGACY AND DYNAMICS IN THOMPSON'S GROUPS" by Belk and Matucci https://arxiv.org/pdf/0708.4250.pdf
- "On the construction of knots and links from Thompson's Groups" by Vaughan Jones, https://arxiv.org/abs/1810.06034