Knots and Graphs

Working Group [Summer 2021]

MATH 4193, class number 16336

Instructor: ***Sergei Chmutov***

RESEARCH PROJECTS

Project 1. ***Virtual links and arrow polynomial.*** (Joseph Paugh, Justin Wu, Gavin Zhang)

This is a project about virtual links. Arrow polynomial of virtual links was introduced by H. Dye and L. Kauffman in [DK]. Later on H. Dye introduced [Dye] a concept of cut points which was used in [De, DJK]. for checkerboard colorable virtual link diagrams. From the other hand the checkerboard colorable diagram of virtual links were used in [CK] and [Ch, Sec.6.8] to produced a Thistlethwaite type theorem for virtual links. The goal of the project is to study the concept of Dye's cut points and trying to use it for Thistlethwaite type theorems. In particular, how to construct a ribbon graph from a virtual link diagram with cut points so that the the Bollobás–Riordan polynomial of the graph would be specialized to the arrow polynomial.

Project 2. ***Thompson index.*** (Torey Hilbert, Alexander Patterson, John White)

This is a development of the last year project.

A Thompson group F is a group of piecewise linear homeomorphism of $[0, 1]$ with breaks at dyadic rational numbers and such that on each interval of linearity its derivatives are powers of 2. (see [CF, CFP]). Every element of F can be encoded by a pair of binary trees with the same number of vertices. V. Jones [Jo] suggested a way to construct a link for every such element and proved that every link can be constructed in this way. The *Thompson index* of a link L is the minimal number of leaves on the pair of trees needed to represent L in the Jones construction. The aim of the project is to start classification of knots and liks by the Thompson index. That is to calculate the Thompson index of first knots and links. For this we probably need to use, and perhaps modify, the program written by Dennis Sweeney last year [Swe].

Project 3. ***Spatial graphs.*** (Aditya Jambhale, Stephen Forest, James Longo)

In a famous paper [CG] John Conway and Cameron Gordon showed that every embedding of the complete graph K_6 in $\mathbb{R}^3$ contains at least one pair of linked triangles and every embedding of K_7 contains a non-trivial knot. An excellent exposition of this results see in [Ad, Ch.8]. A paper [FMMNN] is a survey of this area.

A key point of the proof of the Conway–Gordon theorem was to show the sum of the linking numbers of two disjoint triangles of K_6 is always odd. From the other hand, in [KK] it was shown that such sum for K_n with $n > 6$ is always even.

The aim of the project is to try to generalize the results of [KK]. One such generalization could be to consider the Casson invariant of a (Hamiltonian) knots in K_n. Is it true that is is always even for $n > 7$? Another one could be to consider instead of the linking number or the Casson invariant some higher order invariants, say the Jones, or Conway polynomials. Is it possible to say something general for their sums? The third generalization could be to sum up not all possible two component links but only those with equal number of vertices for even values of n. The forth generalization could be to try to find the smallest value of n for which any embedding of K_n has to contain a Borromean ring. For that perhaps instead of linking numbers one should consider Milnor's triple number.

Project 4. ***Acyclic orientations of signed graphs.*** (Oscar Coppola, Michael Reilly)

This is a continuation of the last year project.

A classical Stanley's theorem claims that the evaluation of the chromatic polynomial at -1 is equal to the number of acyclic orientations of the graph. It has a generalization [St1, Theorem 3.3] to the symmetric chromatic function from Project 1.

From the other hand, there is a generalization [Za] of the chromatic polynomial to *signed graphs*, where a sign ± 1 is assigned to every edge of a graph. Zaslavsky also generalized [Za, Theorem 3.5] Stanley's acyclicity theorem to signed graphs.

Stalney's chromatic symmetric function was generalized to signed graphs in the previous years programs (see slides of presentations [Ra, Ch]). The goal of the project is to find a symmetric generalization of [Za, Theorem 3.5] which would be an analog of [St1, Theorem 3.3].

Last year we got some very essential results in this project [FHR, HHJ] and hopefully we will finish it this year.

References

[Ad] C. Adams, *The knot book*, AMS, 2001.

[CF] J. W. Cannon, W. J. Floyd, *What is ... Thompson's Group?*, Notices AMS **58**(8) (2011) 1112–1113.

[CFP] J. W. Cannon, W. J. Floyd, W. R. Parry, *Introductory notes on Richard Thompson's groups*, L'Enseignement Mathématique **42** (1996) 215–256.

[CK] S. Chmutov, I. Pak, *The Kauffman bracket of virtual links and the Bollobás-Riordan polynomial*, Moscow Mathematical Journal **7**(3) (2007) 409–418. Preprint `arXiv:math.GT/0609012`.

[Ch] S. Chmutov, *Generalized duality for graphs on surfaces and the signed Bollobás-Riordan polynomial*, Journal of Combinatorial Theory, Ser. B **99**(3) (2009) 617–638. Preprint `arXiv:math.CO/0711.3490`.

[CG] J. Conway, C. Gordon, *Knots and links in spatial graphs*, J. Graph Theory, **7** (1983) 445–453.

[DJK] Q. Deng, X. Jin, L. Kauffman, *On arrow polynomials of checkerboard colorable virtual links*, Preprint `arXiv:2002.07361v1 [math.GT]`

[De] Q. Deng, *One conjecture on cut points of virtual links and the arrow polynomial of twisted links*, Preprint `arXiv:2103.12283v2 [math.GT]`

[DK] H. Dye, L. Kauffman, *Virtual Crossing Number and the Arrow Polynomial*, J. Knot Theory Ramifications, **18**(10)(2009) 1335–1357. Preprint `arXiv:math.GT/0810.3858`

[Dye] H. Dye, *Cut points: An invariant of virtual links*, J. Knot Theory and Its Ramifications **26**(9) (2017) ArticleID: 1743006 (10 pages).

[FHR] E. Fawcett, T. Hilbert, M. Reilly, *Covering Graphs and Linear Extensions of Signed Posets.* `https://people.math.osu.edu/chmutov.1/wor-gr-su20/Fawcett-Hilbert-Reilly-slides.pdf`

[FMMNN] E. Flapan, T. Mattman, B. Mellor, R. Naimi, R. Nikkuni, *Recent Developments in Spatial Graph Theory*, Preprint `arXiv: 1602.08122v2 [math.GT]`.

[HHJ] J. Huryn, K. Husar, H. Johnson, *Signed posets and a B-symmetric generalization of Stanley's acyclicity theorem*. `https://people.math.osu.edu/chmutov.1/wor-gr-su20/Huryn-Husar-Johnson-slides.pdf`

[Jo] V. Jones, *On the construction of knots and links from Thompson's groups*. Preprint `arXiv:1810.06034` (2018).

[KK] A. Kazakov, Ph. Korablev, *Triviality of the Conway–Gordon function* ω_2 *for spatial complete graphs*, Journal of Mathematical Sciences, **203**(4) (2014) 490–498.

[Ch] S. Chmutov, *B-symmetric chromatic function of signed graphs*, `https://people.math.osu.edu/chmutov.1/talks/2020/slides-Moscow.pdf` Video is available at `https://www.youtube.com/watch?v=khA7rP84sYY`

[Ra] R. Raghavan, *A Symmetric Chromatic Function for Signed Graphs*, `https://people.math.osu.edu/chmutov.1/wor-gr-su19/Rushil-Raghavan-MIGHTY_10192019.pdf`

[St1] R. Stanley, *A symmetric function generalization of the chromatic polynomial of a graph*, Advances in Math. **111**(1) (1995) 166–194.

[Swe] D. Sweeney, *Programs for Thompson's links*, `https://github.com/sweeneyde/thompson_knots`

[Za] T. Zaslavsky, *Signed graph coloring*, Discrete Mathematics **39**(2) (1982) 215–228.