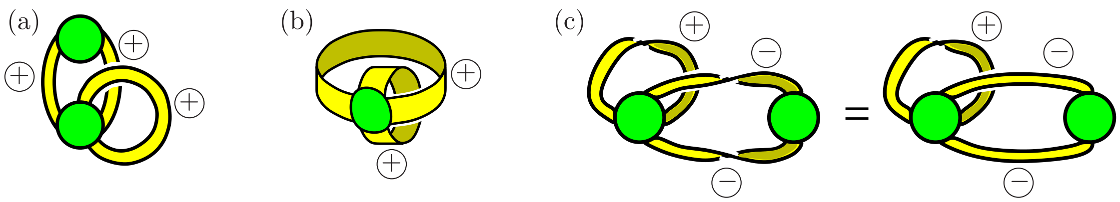
Ribbon graphs

Definition. A *ribbon graph* G is a surface (possibly non-orientable) with boundary, represented as the union of two sets of closed topological discs called *vertices* $V(G)$ and *edges* $E(G)$, satisfying the following conditions:

- these vertices and edges intersect by disjoint line segments;
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.



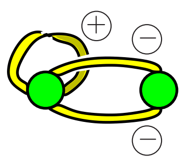
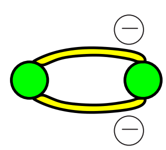
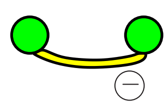
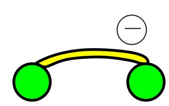
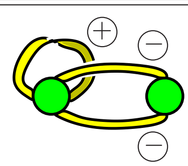
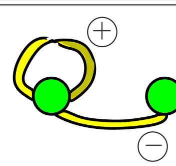
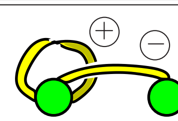
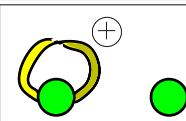
The Bollobás-Riordan polynomial

Reference: B. Bollobás and O. Riordan [BR].

$$R_G(\{x_e, y_e\}, X, Y, Z) := \sum_{F \subseteq G} \Big(\prod_{e \in F} x_e\Big)\Big(\prod_{e \notin F} y_e\Big) X^{r(G)-r(F)} Y^{n(F)} Z^{k(F)-\mathrm{bc}(F)+n(F)}$$

For signed graphs, we set $\begin{cases} x_+ = 1, & x_- = (X/Y)^{1/2}, \\ y_+ = 1, & y_- = (Y/X)^{1/2}. \end{cases}$

Example.

				
(k,r,n,bc) term of R_G	(1,1,1,2) X	(1,1,0,1) 1	(1,1,0,1) 1	(2,0,0,2) Y
				
	(1,1,2,1) XYZ^2	(1,1,1,1) YZ	(1,1,1,1) YZ	(2,0,1,2) Y^2Z

$$R_G(X,Y,Z) = X + 2 + Y + XYZ^2 + 2YZ + Y^2Z$$

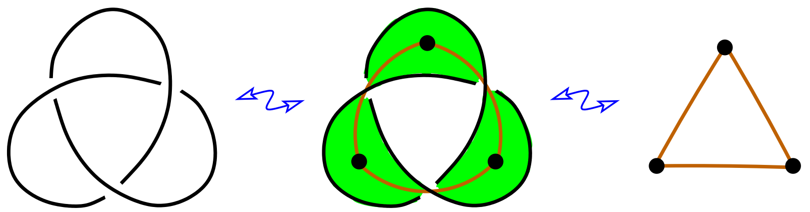
Properties.

$R_G = x_e R_{G/e} + y_e R_{G-e}$ if e is ordinary, that is neither a bridge nor a loop,

$R_G = (x_e + X y_e) R_{G/e}$ if e is a bridge.

$R_{G_1 \sqcup G_2} = R_{G_1 \cdot G_2} = R_{G_1} \cdot R_{G_2}$

Thistlethwaite's Theorem [Ka1] *Up to a sign and multiplication by a power of t the Jones polynomial $J_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_\Gamma(-t,-t^{-1})$.*



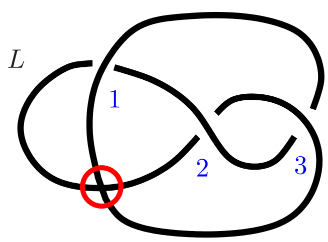
The theorem was generalized to non-alternating links using signed graphs in [Ka2] and using the Bollobás-Riordan polynomial for ribbon graphs in [DFKLS]; and to virtual links in [ChVo, Ch].

Theorem [Ch].

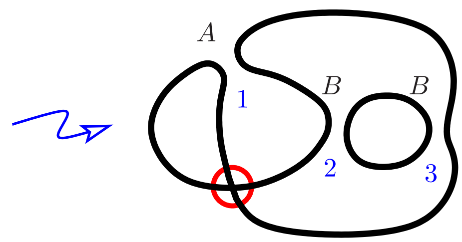
Let L be a virtual link diagram with e classical crossings, G^s_L be the signed ribbon graph corresponding to a state s, and $v := v(G^s_L)$, $k := k(G^s_L)$. Then $e = e(G^s_L)$ and

$$[L](A,B,d) = A^e \left(X^k Y^v Z^{v+1} R_{G^s_L}(X,Y,Z) \Big|_{X=\frac{Ad}{B},\ Y=\frac{Bd}{A},\ Z=\frac{1}{d}} \right) .$$

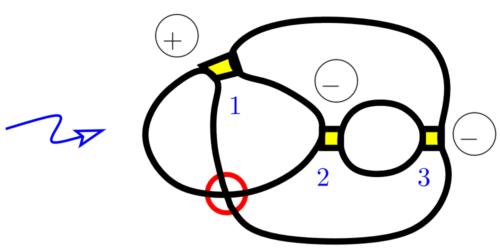
Construction of a ribbon graph from a virtual link diagram



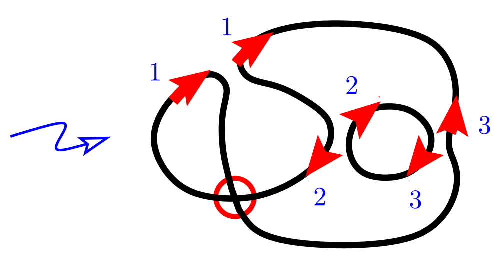
Diagram



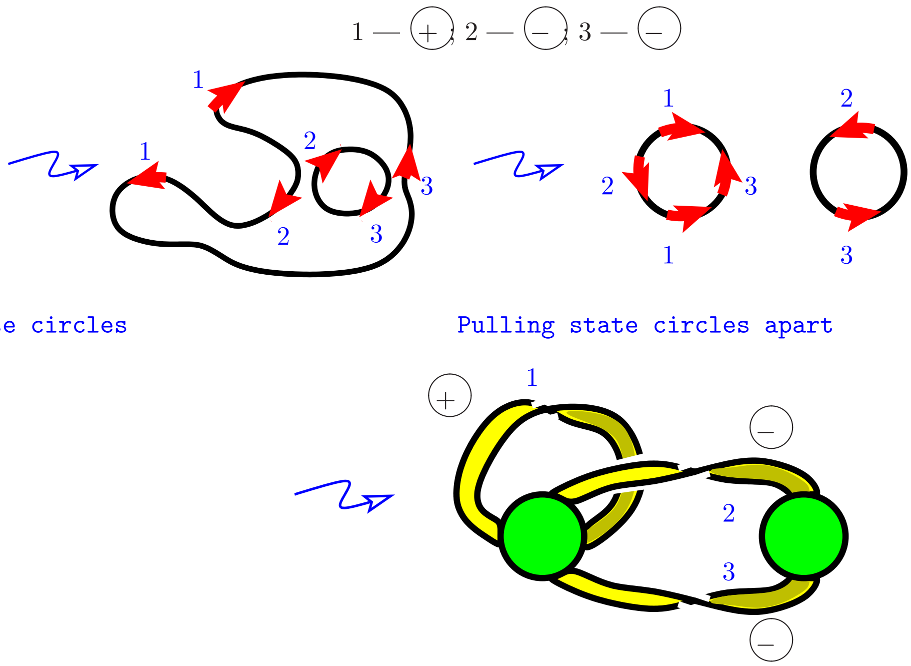
State s



Attaching planar bands



Replacing bands by arrows



Untwisting state circles

Pulling state circles apart

Forming the ribbon graph G_L^s

References

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[ChVo] S. Chmutov, J. Voltz, *Thistlethwaite's theorem for virtual links*. Journal of Knot Theory and Its Ramifications, **17**(10) (2008) 1189-1198; preprint `arXiv:math.GT/0704.1310`.

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[Ka1] L. H. Kauffman, *New invariants in knot theory*, Amer. Math. Monthly **95** (1988) 195–242.

[Ka2] L. H. Kauffman, *A Tutte polynomial for signed graphs*, Discrete Appl. Math. **25** (1989) 105–127.