Symmetric functions. [Eg]

Definition. A *symmetric function* on n (may be infinitely many) variables $x_1, \dots, x_n$ is a function that is unchanged by any permutation of its variables. If π is a permutation of $\{1, 2, \dots, n\}$, then for a symmetric function $f(x_1, \dots, x_n)$ we have $f(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$ So it is invariant under the action of the *symmetric group* S_n acting on the indices of variables.

The symmetric functions form a *vector space* $\mathbb{R}[x_1, \dots, x_n]^{S_n}$. Moreover, they form an *algebra* (a vector space with an operation of multiplication of vectors).

Elementary symmetric functions.

$$
\begin{aligned}
e_0 &:= 1; \\
e_1 &:= x_1 + x_2 + \cdots + x_n; \\
e_2 &:= (x_1x_2 + x_1x_3 + \cdots + x_1x_n) + (x_2x_3 + \cdots + x_2x_n) + \cdots + x_{n-1}x_n; \\
e_3 &:= x_1x_2x_3 + \ldots \cdots + x_{n-2}x_{n-1}x_n; \\
&\vdots \\
e_n &:= x_1x_2 \ldots x_{n-1}x_n.
\end{aligned}
$$

In general
$$\boxed{e_k := \sum_{1 \le j_1 < j_2 \ldots j_k \le n} x_{j_1} \ldots x_{j_k}}.$$

Vieta's formulas: the elementary symmetric polynomials in $x_1, \dots, x_n$ are coefficients (up to the sign) of a polynomial $p(x)$ with roots $x_1, \dots, x_n$:

$$p(x) = x^n - e_1x^{n-1} + e_2x^{n-2} - \cdots \pm e_{n-1}x \mp e_n = (x - x_1)(x - x_2)\ldots(x - x_n).$$

Fundamental theorem of symmetric polynomials.
Any symmetric polynomial $p(x_1, \dots, x_n)$ can be uniquely(!!!) represented as a polynomial in the elementary symmetric functions: $p(x_1, \dots, x_n) = q(e_1, \dots, e_n)$ for an appropriate polynomial $q(y_1, \dots, y_n)$.

In other words, the algebra of symmetric polynomials $\mathbb{R}[x_1, \dots, x_n]^{S_n}$ is isomorphic to the algebra of polynomials $\mathbb{R}[y_1, \dots, y_n]$. Thus, the elementary symmetric polynomials $e_1, \dots, e_n$ constitute the *generating set* for the algebra $\mathbb{R}[x_1, \dots, x_n]^{S_n}$. As a vector space, it has a basis (linear, or additive) consisting of all monomials in $e_1, \dots, e_n$.

Symmetric power functions

$$\boxed{p_k(x_1, \dots, x_n) := x_1^k + x_2^k + \cdots + x_n^k}$$

also form a generating set for $\mathbb{R}[x_1, \dots, x_n]^{S_n}$.

Newton's identities.

$$ke_k = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i} p_i.$$

In particular, $2e_2 = e_1p_1 - e_0p_2 = e_1^2 - p_2$ because $p_1 = e_1$ and $e_0 = 1$. So, $p_2 = e_1^2 - 2e_2$.

Stanley's chromatic symmetric function. [St1]

$$\boxed{X_G(x_1, x_2, \dots) := \sum_{\substack{\varkappa: V(G) \to \mathbb{N} \\ \text{proper}}} \prod_{v \in V(G)} x_{\varkappa(v)}}$$

For example, $\chi_G(q) = X_G(\underbrace{1, 1, \dots, 1}_{q}, 0, 0, 0, \dots)$.

Example.

X $= \widehat{x_1x_1} + x_1x_2 + x_1x_3 + \dots$
$x_2x_1 + \widehat{x_2x_2} + x_2x_3 + \dots$
$x_3x_1 + x_3x_2 + \widehat{x_3x_3} + \dots$
$\vdots \quad \vdots \quad \ddots$

$= p_1^2 - p_2 = e_1^2 - (e_1^2 - 2e_2) = 2e_2,$ where $p_m := \sum_{i=1}^{\infty} x_i^m$ *is the power function basis* for the space of symmetric functions. In general, $X_{K_k} = k!e_k$.

Symmetric Stanley's acyclicity theorem deals with the expression of X_G in terms of *elementary symmetric functions.*

Theorem. [St1, Theorem 3.3] *Let* $X_G = \sum c_{l_1,l_2,\dots,l_s} e_{l_1} e_{l_2} \dots e_{l_s}$ *be the expression of* X_G *in terms of elementary symmetric functions. (Note that* $l_1 + l_2 + \dots + l_s = \#$ *of vertices of* G*.) Then for every* s, $\sum c_{l_1,l_2,\dots,l_s} = \#$ *of acyclic orientations of* G *with exactly* s *sinks.*

For example, the graph $G =$ has 2 acyclic orientations with exactly one sink  and , because X $= 2e_2$.

Chromatic polynomial of signed graphs.

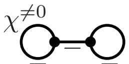
There are two chromatic polynomials of signed graphs [Za].

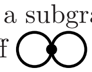
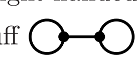
A q-*coloring* of a signed Γ is a map $\varkappa : V(\Gamma) \to \{-q, -q+1, \dots, -1, 0, 1, \dots, q-1, q\}$. A q-coloring $\varkappa$ is *proper* if for any edge e with the sign ε_e: $\varkappa(v_1) \neq \varepsilon\varkappa(v_2)$, where v_1 and v_2 are the endpoints of e.

Definition.
$\chi_\Gamma(2q+1) := \#$ *of proper* q*-colorings of* Γ.
$\chi_\Gamma^{\neq 0}(2q) := \#$ *of proper* q*-colorings of* Γ *which take nonzero values.*

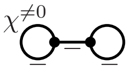
Properties.

- $\chi_\Gamma(l)$ is a polynomial function of $l = 2q+1 > 0$;
- $\chi_\Gamma^{\neq 0}(l)$ is a polynomial function of $l = 2q > 0$;
- $\chi_\Gamma(l) = \chi_{\Gamma - e}(l) - \chi_{\Gamma/e}(l)$;
- $\chi_\Gamma^{\neq 0}(l) = \chi_{\Gamma - e}^{\neq 0}(l) - \chi_{\Gamma/e}^{\neq 0}(l)$;
- $\chi_{\Gamma_1 \sqcup \Gamma_2} = \chi_{\Gamma_1} \cdot \chi_{\Gamma_2}$ and $\chi_{\Gamma_1 \sqcup \Gamma_2}^{\neq 0} = \chi_{\Gamma_1}^{\neq 0} \cdot \chi_{\Gamma_2}^{\neq 0}$ for a disjoint union $\Gamma_1 \sqcup \Gamma_2$;
- $\chi_\emptyset = 1$.

Example. $\chi^{\neq 0}$ $(2q) = 2q(2q-1)$.

There are two tricky issues in Zaslavky's acyclicity theorem. The first one is the notion of a cycle. A subset S of edges of a sign graphs is called *balanced* if for every circuit in S the product of the signs of edges of the circuit is equal to 1. A *cycle* of a signed graph G is a subgraph of one the following 3 types: 1) a balanced circuit, 2) a subdivision of a tight handcuff  with both circuits to be unbalanced, and 3) a subdivision of a loose handcuff  with both circuits to be unbalanced. The second issue is a notion of orientation. An *orientation* of an edge is a pair of arrows on its half-edges which are coherent for positive edges and not coherent for negative edges. An orientation of a sign graph is *acyclic* if every cycle contains either a source or a sink. An orientation of a sign graph is *compatible* with a coloring c if for every positive edge the color of it arrow-head is greater or equal to the color of it arrow-tail and for every negative edge the sum of colors of its ends is not negative (resp. not positive) for the arrows pointed towards the ends (resp. away from the end).

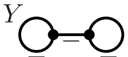
Theorem. [Za, Theorem 3.5] *Let $q \in \mathbb{N}$ and G be a signed graph with n vertices. The the number of compatible pairs of acyclic orientations of G and colorings $V(G) \to \{-q, -q+1, \ldots, -1, 1, \ldots, q-1, q\}$ is equal to $(-1)^n \chi_G^{\neq 0}(-2q)$.*

Example. For $q = 1$ and the graph above we have $\chi^{\neq 0}$$(-2) = 6$. Here are 6 compatible pairs of acyclic orientations and colorings $V(G) \to \{-1, 1\}$ (we mark the source-vertex red and the sink-vertex blue).

Note that the first two colorings are proper and the last four are improper.

***C*-symmetric chromatic function of signed graphs**. [Ra, Ch]

$$Y_G(\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots\ldots) := \sum_{\substack{\varkappa: V(G) \to \mathbb{Z}\setminus\{0\} \\ \text{proper}}} \prod_{v \in V(G)} x_{\varkappa(v)}$$

Example. Y
$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_{-2}(\ldots x_{-2} + x_{-1} + x_1 + \widehat{x_2} + \ldots)$$
$$x_{-1}(\ldots x_{-2} + x_{-1} + \widehat{x_1} + x_2 + \ldots)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$= p_{1,0}^2 - p_{1,1}, \text{ where}$$

$p_{a,b} := \sum_{i \in \mathbb{Z}\setminus\{0\}} x_i^a x_{-i}^b$ are the *signed power functions*.

The signed chromatic polynomial $\chi_G^{\neq 0}(2q)$ is a specialization of Y_G obtained by substitution $x_i = 1$ for $|i| \le q$ and $x_i = 0$ for $|i| \ge q$. This is the same substitution as $p_{a,b} = l = 2q$.

References

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