Δ-*matroids* [Bouchet]

Matroids	Δ-matroids
A matroid is a pair $M = (E, \mathcal{B})$ consisting of a finite set E and a nonempty collection $\mathcal{B}$ of its subsets, called *bases*, satisfying the axioms: (**B1**) *No proper subset of a base is a base.* (**B2**) =(**Exchange axion**) *If B_1 and B_2 are bases and $b_1 \in B_1 - B_2$, then there is an element $b_2 \in B_2 - B_1$ such that $(B_1 - b_1) \cup b_2$ is a base.*	*A Δ-matroid* is a pair $M = (E, \mathcal{F})$ consisting of a finite set E and a nonempty collection $\mathcal{F}$ of its subsets, called *feasible sets*, satisfying the **Symmetric Exchange axion** *If F_1 and F_2 are two feasible sets and $f_1 \in F_1 \Delta F_2$, then there is an element $f_2 \in F_1 \Delta F_2$ such that $F_1 \Delta \{f_1, f_2\}$ is a feasible set.*

Ribbon graphs (graphs on surfaces)

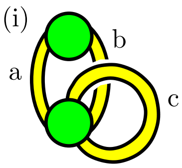
Definition. A *ribbon graph* G is a surface (possibly non-orientable) with boundary, represented as the union of two sets of closed topological discs called *vertices* $V(G)$ and *edges* $E(G)$, satisfying the following conditions:

- these vertices and edges intersect by disjoint line segments;
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.

Definition. A *quasi-tree* is ribbon graph G with a single boundary component, $bc(G) = 1$.

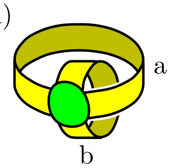
Examples.

(i)



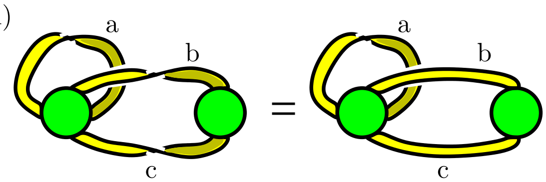
Spanning quasi-trees: $\{a\}, \{b\}, \{a, b, c\}$

(ii)



Spanning quasi-trees: $\emptyset, \{a, b\}$

(iii)



Spanning quasi-trees: $\{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}$

Theorem. *Let $G = (V, E)$ be a ribbon graph. Then $D(G) := (E, \{$spanning quasi-trees$\})$ is a Δ-matroid.*

Minors in Δ-matroids

Let $D = (E, \mathcal{F})$ be a Δ-matroid and $e \in E$.
e is a *loop* iff $\forall F \in \mathcal{F}, e \notin F$. e is a *coloop* iff $\forall F \in \mathcal{F}, e \in F$.
If e is not a loop, $D/e := (E \setminus \{e\}, \{F \setminus \{e\} | F \in \mathcal{F}, e \in F\})$.
If e is not a coloop, $D \setminus e := (E \setminus \{e\}, \{F | F \in \mathcal{F}, F \subset E \setminus \{e\}\})$.

Twists of Δ-matroids. Let $D = (E, \mathcal{F})$ be a Δ-matroids and $A \subset E$.

$$D * A := (E, \{F \Delta A | F \in \mathcal{F}\}).$$

Dual Δ-matroid: $D^* := D * E$.

Matroids associated with a Δ-matroid

Let $D=(E,\mathcal{F})$ be a Δ-matroid.
$D_{min}:=(E,\mathcal{F}_{min})$, where $\mathcal{F}_{min}:=\{F\in\mathcal{F}|F$ is of minimal possible cardinality$\}$.
$D_{max}:=(E,\mathcal{F}_{max})$, where $\mathcal{F}_{max}:=\{F\in\mathcal{F}|F$ is of maximal possible cardinality$\}$.

Facts.

- D_{min} and D_{max} are usual matroids. *Width* $w(D):=r(D_{max})-r(D_{min})$.
- $(D(G))_{min}=\mathcal{C}(G)$. $(D(G))_{max}=(\mathcal{C}(G^*))^*$.
- $D(G)=\mathcal{C}(G)$ iff G is a planar ribbon graph.

The twist polynomial of a delta-matroid $D=(E,\mathcal{F})$

is the generating function for the width of all twists of D,

$$^{\partial}w_D(z):=\sum_{A\subseteq E} z^{w(D*A)}$$

References

[Bouchet] A. Bouchet, *Greedy algorithm and symmetric matroids*, Math. Program. **38** (1987) 147–159.