The Arrow Polynomial

Recall that the Jones polynomial of an oriented knot $\vec{K}$ is given by

$$J_{\vec{K}}(A) = (-A)^{-3w(\vec{K})} \sum_{S \in \mathcal{S}(\vec{K})} A^{\alpha(S)} B^{\beta(S)} d^{\delta(S)-1}$$

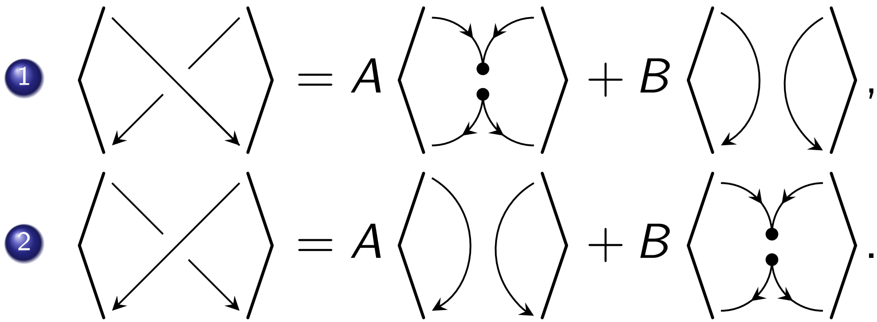
where $B = A^{-1}$ and $d = -A^2 - B^2$. Likewise, we define the arrow polynomial

$$\mathcal{A}_{\vec{K}}(A, K_1, K_2, ...) = (-A)^{-3w(\vec{K})} \sum_{S \in \mathcal{S}(\vec{K})} A^{\alpha(S)} B^{\beta(S)} d^{\delta(S)-1} \langle S \rangle$$

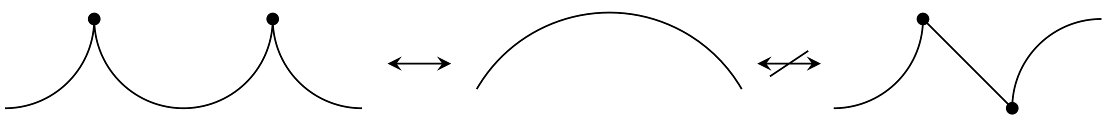
where $\langle S \rangle$ is called the *evaluation* of the state (to be defined). Note that $\mathcal{A}_{\vec{K}} \in \mathbb{Z}[A, A^{-1}, K_1, K_2, ...]$ with all the variables commuting and independent.

State Evaluation

We define the oriented state expansion by



In contrast to the usual state reduction, we now have cusps on each of the remaining circles. Next, we bring each loop into its *reduced state* by "canceling" similar adjacent cusps:



State Evaluation Continued

At this point, each loop C in the state is reduced and has $2n$ cusps. Now we let

$$\langle C \rangle = \begin{cases} 1 & \text{if } n = 0 \\ K_n & \text{otherwise} \end{cases}.$$

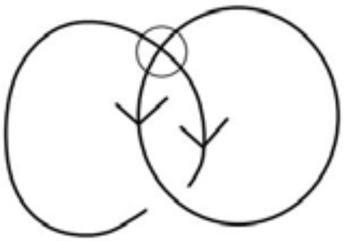
Finally, we define

$$\langle S \rangle = \prod_C \langle C \rangle.$$

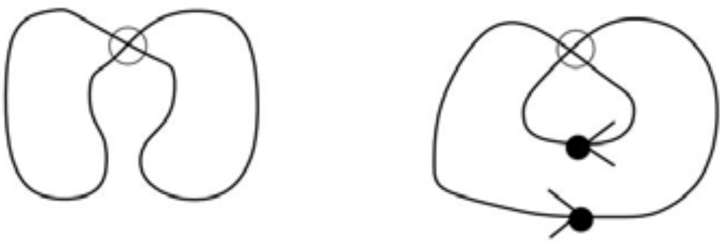
This turns out to be invariant under classical and virtual Reidermaster moves.

Arrow Polynomial of the Virtual Hopf Link

Let $\vec{K}$ denote the knot



Its states are

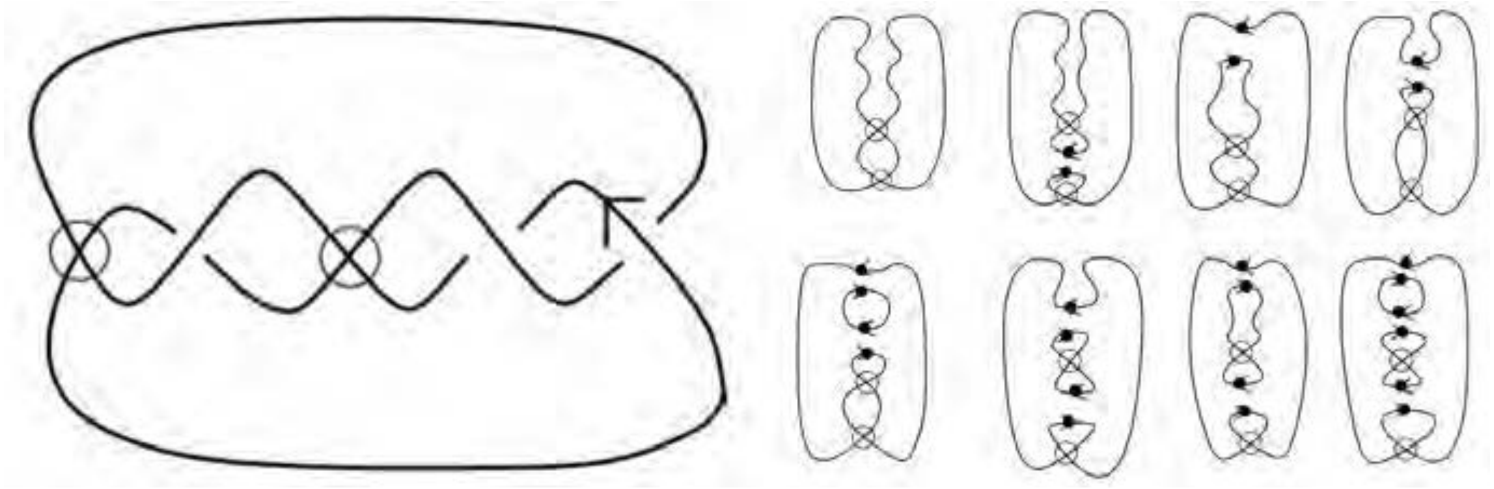


Hence, $\langle S_1 \rangle = B$ and $\langle S_2 \rangle = AK_1$. Thus,

$$\mathcal{A}_{\vec{K}} = -A^{-3}(A^{-1} + K_1 A).$$

Virtualized Trefoil

Consider the knot $\vec{K}$ with its states:



For example, we have $\langle S_6 \rangle = A^2 B d K_1^2$ and $\langle S_8 \rangle = A B^2 d^2 K_1^2$. It turns out that

$$\mathcal{A}_{\vec{K}} = -A^{-3}(-A^{-5} + K_1^2 A^{-5} - K_1^2 A^3).$$

Twisted Arrow Polynomial

Recall that given an oriented twisted knot $\vec{\mathcal{K}}$, we define the *twisted Jones polynomial* of an oriented twisted knot $\vec{\mathcal{K}}$ by

$$\mathscr{J}_{\vec{\mathcal{K}}}(A, M) = (-A)^{-3w(\vec{\mathcal{K}})} \sum_{S \in \mathcal{S}(\vec{\mathcal{K}})} A^{\alpha(S)} B^{\beta(S)} d^{\gamma_1(S)} M^{\gamma_2(S)}$$

where

- $\gamma_1(S)$ is the number of circles with an even number of bars, and
- $\gamma_2(S)$ is the number of circles with an odd number of bars.

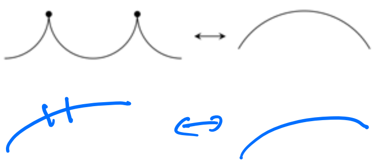
So naturally, we define the twisted arrow polynomial as

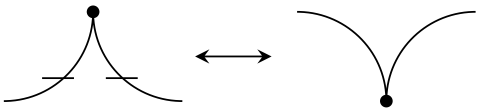
$$\mathscr{A}_{\vec{\mathcal{K}}}(A, M, K_1, K_2, \ldots) = (-A)^{-3w(\vec{\mathcal{K}})} \sum_{S \in \mathcal{S}(\vec{\mathcal{K}})} A^{\alpha(S)} B^{\beta(S)} d^{\delta(S)-1} \langle S \rangle .$$

However, we add a few new rules for evaluating the state.

Twisted State Evaluation

In addition to the reduction rules from before, we also stipulate that



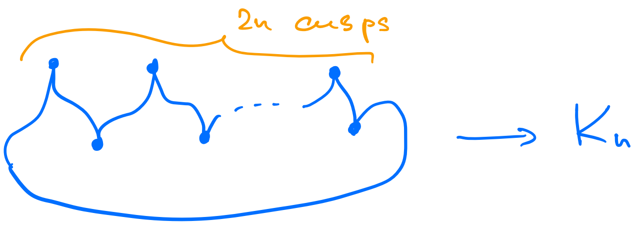


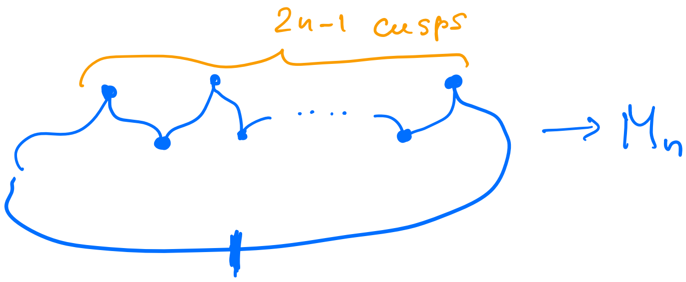
With this, we can reduce each loop in the state. We now define $\langle C \rangle$ the same as before with a small change. If C is a trivial loop with bar, we let $\langle C \rangle = M$.

Now like before, we define

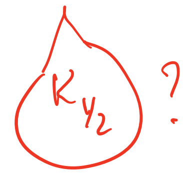
$$\langle S \rangle = \prod_{C} \langle C \rangle .$$

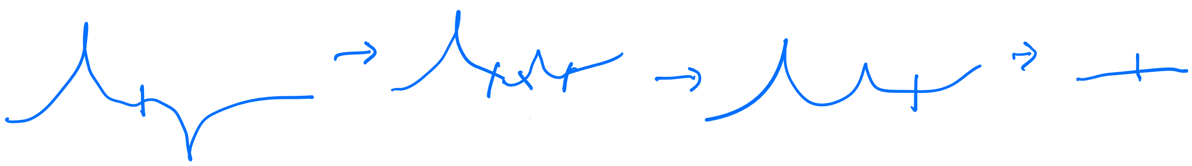
How about this

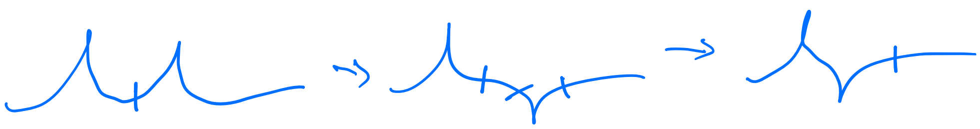


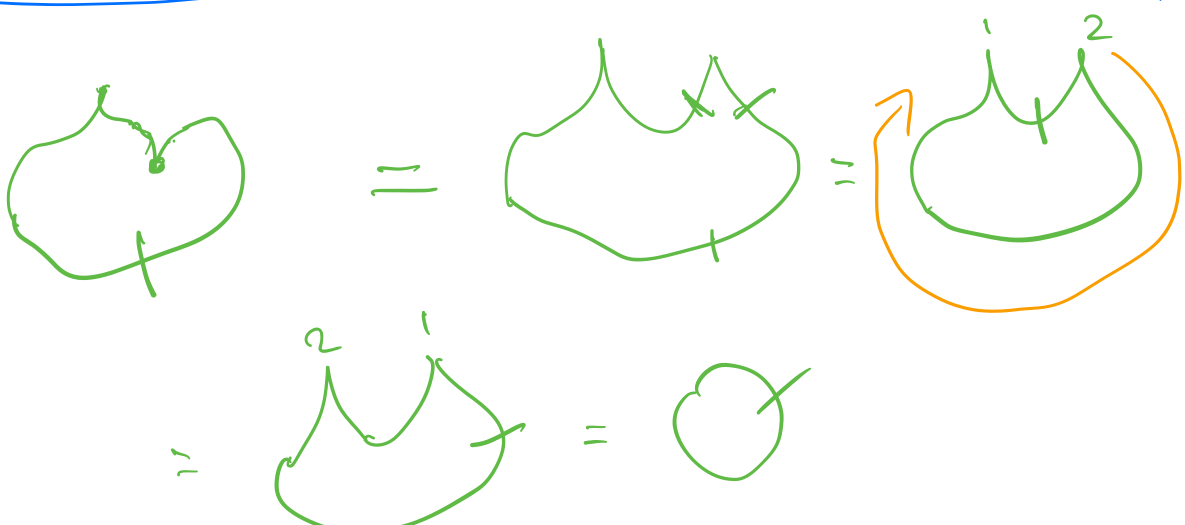


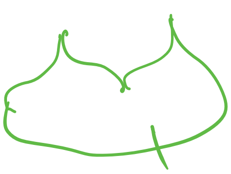
$\mathcal{A}_{\vec{k}}(A, M, K_1, K_2, \ldots, M_1, M_2 \ldots)$











References

1. Virtual Crossing Number and the Arrow Polynomial by H. A. Dye and Louis H. Kauffman
2. ONE CONJECTURE ON CUT POINTS OF VIRTUAL LINKS AND THE ARROW POLYNOMIAL OF TWISTED LINKS by QINGYING DENG