An Excerpt

Let H be a forbidden moves set, we call a subset S of H a generating set, if any other forbidden move of H may be obtained by a finite sequence of extended Reidermaster moves and moves from the set S. If generating set S has the minimum number of generators, we call S a minimum generating set.

If forbidden moves $T_4, T_5, T_6, T_7, T_8, T_9, F_1, F_2, F_3$ and F_4 are allowed, any twisted knot can be deformed into a trivial knot (with a bar). Clearly, set $H = \{T_4, T_5, T_6, T_7, T_8, T_9, F_1, F_2, F_3, F_4\}$ is a generating set of all forbidden moves (contain F_o, F_s, F_u, F_v and other forbidden moves). Theorem 3.1 tells us that up to three forbidden moves T_4, F_1 (or F_2) and F_3 (or F_4) are required, that is, set $S = \{T_4, F_1(\text{or } F_2), F_3(\text{or } F_4)\}$ is a generating set of H. And we conjecture that this result is optimal.

Our problem

Conjecture 3.1

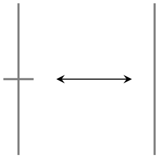
Cardinality of minimum generating set of all forbidden moves is 3. And S contains T_4 and contains one of the pairs $(F_1, F_3), (F_1, F_4), (F_2, F_3)$, and (F_2, F_4).

The two interpretations:

1. H is any forbidden moves set imaginable.
2. $H = \{T_4, T_5, T_6, T_7, T_8, T_9, F_1, F_2, F_3, F_4\}$

Interpretation 1: H is arbitrary

We introduce the following move



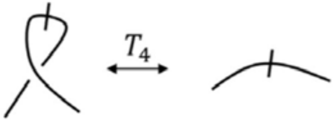
called the *untwist*.

An old paper by Goussarov, Polyak, and Viro showed that any virtual knot can be transformed into the unknot via the forbidden moves F_1 and F_2. Since $F_1 \equiv F_2$ in the twisted theory, the set $\{U, F_1\}$ is enough to turn any twisted knot into the unknot.

Interpretation 2: $H = \{ T_i, F_j \}$

While this produces a non-trivial problem, it leads to questions such as:

1. Why in the world is this an interesting problem?
2. What's so special about H and $S = \{ T_4, F_1, F_3 \}$?
3. The paper introduces move T_4 as a twisted analogue of the first Reidermaster move:



Why is this a more "natural" forbidden move than the untwist?

A Possible Motivation

Recall that one gets welded knot theory when considering the moves $R_1, R_2, R_3, V_1, V_2, V_3, V_4$, and exactly one of F_1 or F_2.

Hence, no strict subset of $\{F_1, F_2\}$ makes everything trivial.

Therefore, if we prove that no strict subset of $S = \{T_4, F_1, F_3\}$ turns everything into the unknot, then we might obtain twisted welded theory.

However, this still doesn't justify why S is a more natural forbidden moves set than $\{U, F_1\}$.

So far, we have been able to resolve 1 out of

$$\binom{|H|-2}{2} + \binom{|H|-2}{1} = 36$$

cases.

The Twisted Jones Polynomial

Recall that the Jones polynomial of an oriented knot $\vec{K}$ is given by

$$J_{\vec{K}}(A) = (-A)^{-3w(\vec{K})} \sum_{S \in \mathcal{S}(\vec{K})} A^{\alpha(S)} B^{\beta(S)} d^{\delta(S)}$$

where $B = A^{-1}$ and $d = -A^2 - B^2$. Somewhat similarly, we define the *twisted Jones polynomial* of an oriented twisted knot $\vec{\mathcal{K}}$ by

$$\mathscr{J}_{\vec{\mathcal{K}}}(A, M) = (-A)^{-3w(\vec{\mathcal{K}})} \sum_{S \in \mathcal{S}(\vec{\mathcal{K}})} A^{\alpha(S)} B^{\beta(S)} d^{\gamma_1(S)} M^{\gamma_2(S)}$$

where

- $\gamma_1(S)$ is the number of circles with an even number of bars, and
- $\gamma_2(S)$ is the number of circles with an odd number of bars.

Note that the polynomial has two variables now instead of one. This is an invariant quantity under extended Reidermaster moves.

Alternative Definition of the Twisted Jones Polynomial

We first define $\langle \mathcal{K} \rangle$ by:

1. $\langle \varnothing \rangle = 1$,
2. $\left\langle \diagup\!\!\!\!\diagdown \right\rangle = A \left\langle)\,(\right\rangle + B \left\langle \asymp \right\rangle$,
3. $\left\langle D \sqcup \bigcirc \right\rangle = d \left\langle D \right\rangle$,
4. $\left\langle D \sqcup \bigcirc \right\rangle = M \left\langle D \right\rangle$

Then, we let

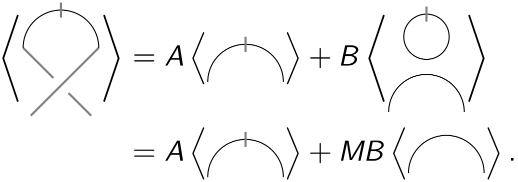
$$\mathscr{J}_{\vec{\mathcal{K}}}(A, M) = (-A)^{-3w(\vec{\mathcal{K}})} \langle \mathcal{K} \rangle .$$

It is now immediate that

$$\mathscr{J}_{\bigcirc}(A, M) = -A^2 - A^{-2} \quad \text{and} \quad \mathscr{J}_{\bigcirc}(A, M) = M.$$

A Small Result

Observe that



Hence, $\mathscr{J}_{\vec{\mathcal{K}}}(1,0)$ is invariant under the T_4 and the extended Reidermaster moves. The knot

has twisted polynomial $\mathscr{J}(A,M) = A^{-6} + (1-M^2)A^{-2}$.Hence, $\mathscr{J}(1,0) = 2$ and so $\{T_4\}$ isn't a generating set.