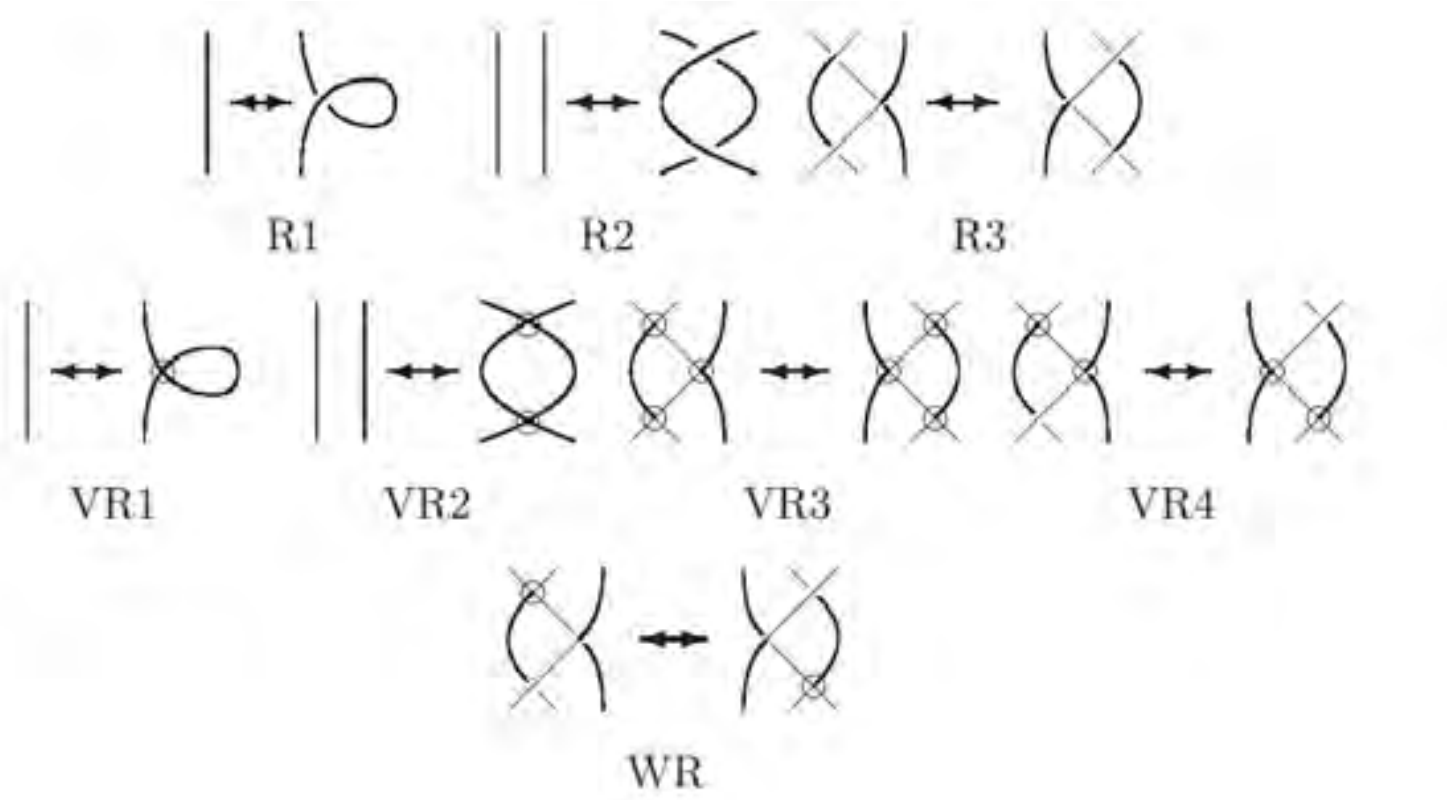
Diagrams Realizing Prescribed Sublink Diagrams

Mark Kikta, Yan Xuan

Knots and Graphs Working Group

July 1, 2022

Virtual and Welded Links



Background

Given a link L of sublinks $L_1,\ldots,L_n$, for any possible diagram D_i of L_i, is there a diagram D of L whose subdiagram $D(L_i)$ of L_i is isotopic to D_i?

Our intuition tells us that as a link diagram becomes more complex, each subdiagram is less likely to be able to be deformed into a prescribed diagram without changing the link.

Classical Case

Our intuition is wrong for classical links.

Theorem 1 (G. T. Jin and J. H. Lee)

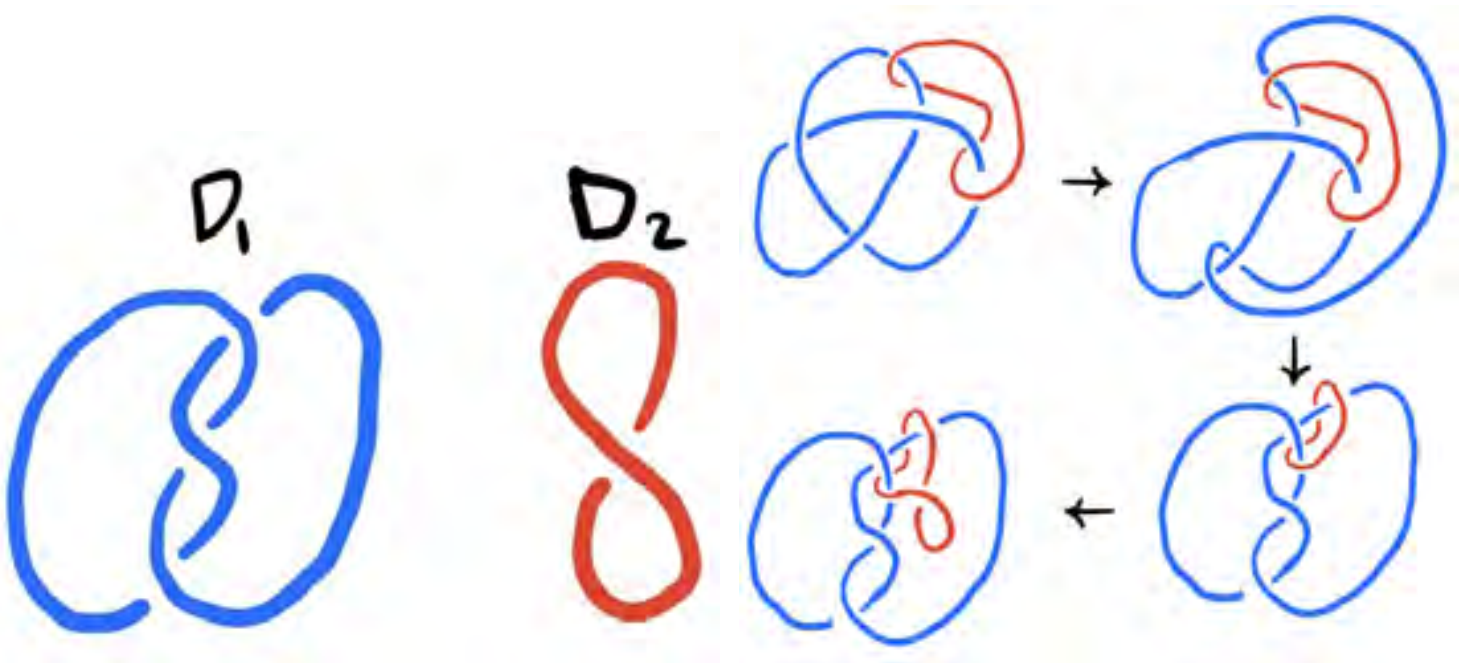
Suppose that $D_1, ..., D_n$ are link diagrams. Given a link L which is partitioned into sublinks $L_1, ..., L_n$ admitting diagrams $D_1, ..., D_n$ respectively, there is a diagram D of L whose restrictions to $L_1, ..., L_n$ are isotopic in $\mathbb{R}^2$ to $D_1, ..., D_n$, respectively.

Remark

We say that L realizes the subdiagrams $D_1, ..., D_n$. Here, $D_1, ..., D_n$ are classical link diagrams.

Example of Theorem 1

Let D_1 and D_2 be the link diagrams on the left, and let L be a link represented by the diagrams on the right. The sequence of steps depicted results in a diagram D of L whose restrictions to L_1 and L_2 are isotopic to D_1 and D_2.



Virtual Case

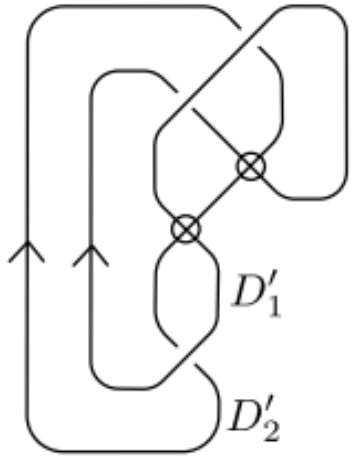
The property does not hold for virtual links, even for those with two components.

Theorem 3 (N. Kamada)

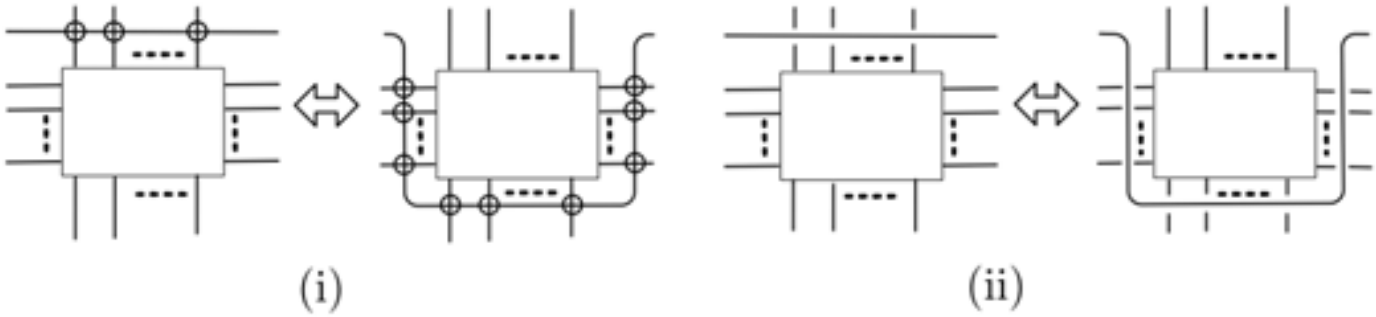
There exist diagrams D_1 and D_2 of virtual links, and a virtual link L which is partitioned into L_1 and L_2 admitting diagrams D_1 and D_2 such that L does not admit any diagram D whose restrictions to L_1 and L_2 are isotopic in $\mathbb{R}^2$ to D_1 and D_2 respectively.

Example of Theorem 3

Let D_1 and D_2 be loops with no crossings, and let the below diagram represent the link L. There is no diagram D of L such that its subdiagrams are isotopic to D_1 and D_2.



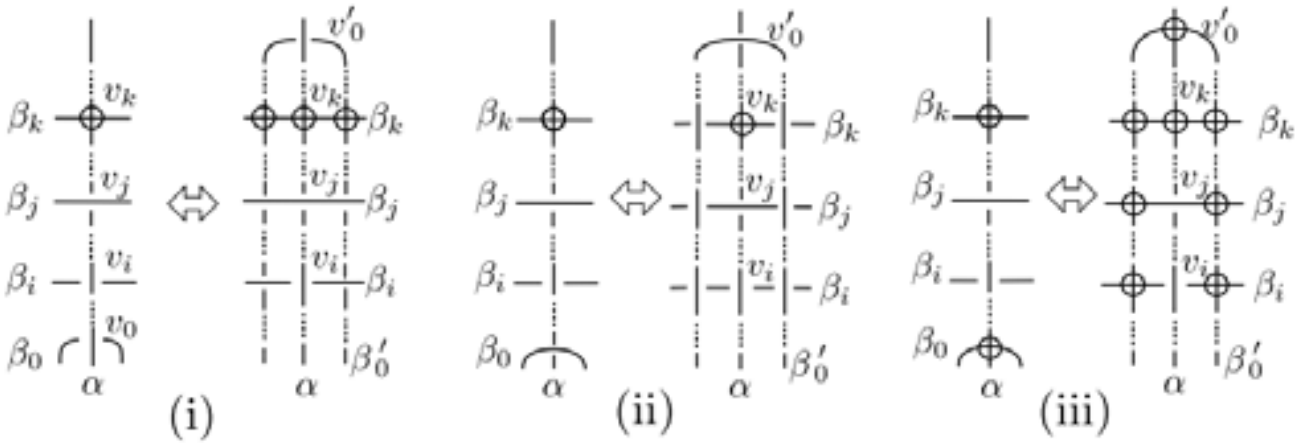
Detour Moves



(i) (virtual) detour move (ii) over detour move

We can define virtual and welded links in terms of two detour moves. Two virtual link diagrams are equivalent if and only if they are related by a sequence of $R1-3$ and (virtual) detour moves. Two welded link diagrams are equivalent if and only if they are related by a sequence of $R1-3$, (virtual) detour moves, and over detour moves. Over detour moves depend on WR.

Finger Moves



Under, over and virtual finger moves

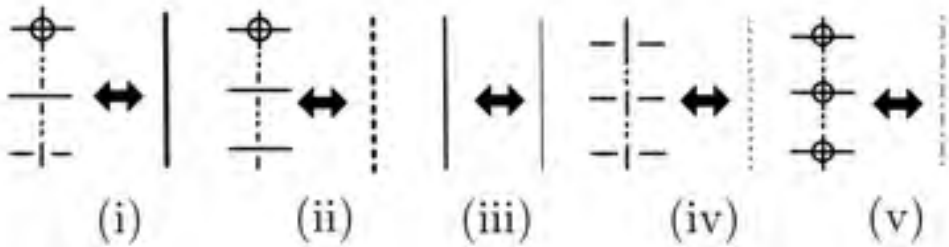
Remark

Under and virtual finger moves may be performed on both virtual and welded links, but over finger moves depend on *WR* and may only be performed on welded links.

Notation

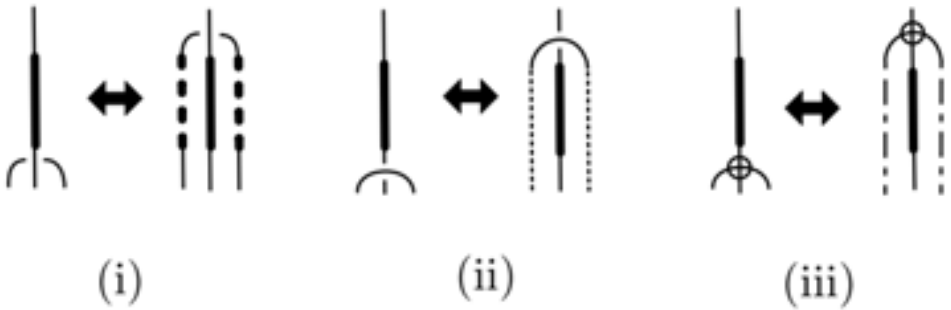
We define notation to record the crossing information for five different types of arcs:

(i) (On a thick line,) no condition is required.
(ii) (On a thick dotted line,) at each classical crossing on γ, the arc γ is an under arc.
(iii) (On a thin line,) there is no crossing on γ.
(iv) (On a thin dotted line,) every crossing on γ is a classical crossing where γ is an over arc.
(v) (On a thin dashed line,) every crossing on γ is a virtual crossing.



Notation

In the following proof, the denoted crossings are mainly between arcs of different sublinks.



Under, over, and virtual finger moves.

All Welded Subdiagrams Are Realizable

(N. Kamada) Theorem 2

Suppose that $D_1, ..., D_n$ are diagrams of welded links. Given a welded link L which is partitioned into sublinks $L_1, ..., L_n$ admitting diagrams $D_1, ..., D_n$ respectively, there is a diagram D' of L whose restrictions to $L_1, ..., L_n$ are isotopic in $\mathbb{R}^2$ to $D_1, ..., D_n$ respectively.

Sketch of the Proof

Any two link diagrams can be obtained from one another by a sequence of local moves. If we can prove that any local move can be performed in any welded link diagram (Lemma 4), then Theorem 2 follows by induction.

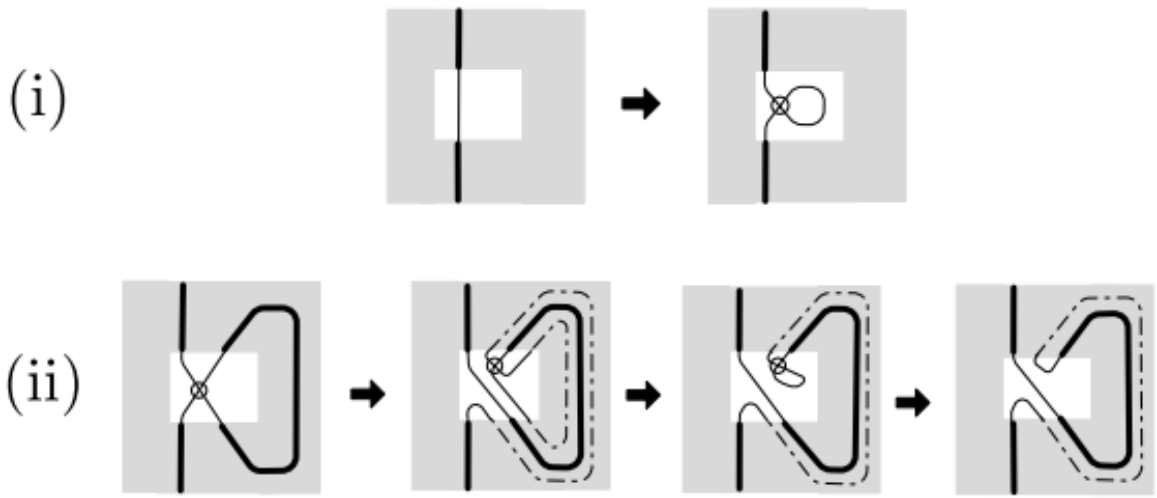
Lemma 4

Let D be a diagram of a welded link L partitioned into L_1 and L_2. Let D_1 be a diagram obtained from $D(L_1)$ by a local R, VR, or WR move. There is a diagram D' of L such that $D'(L_1)$ is isotopic to D_1 in $\mathbb{R}^2$ and $D'(L_2) = D(L_2)$

Notation

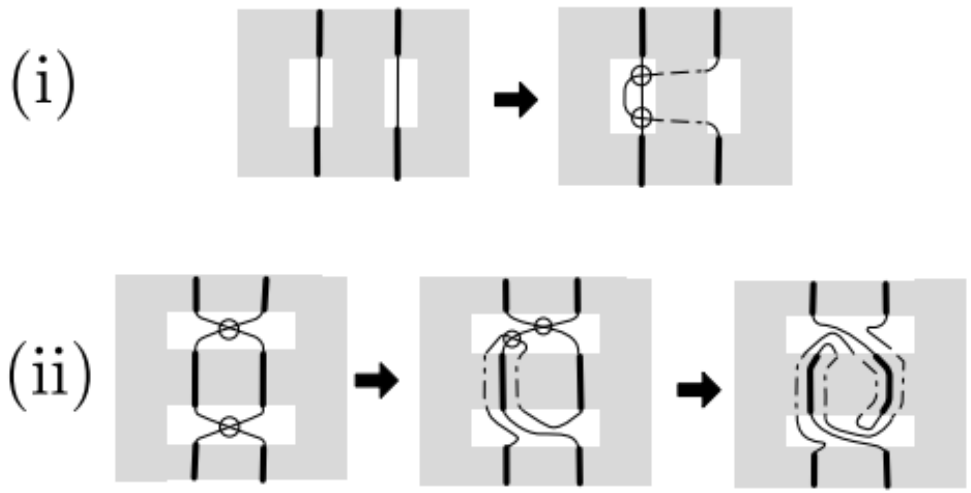
$D(L_1)$ is the subdiagram of D obtained by the restriction of L to L_1.

VR1



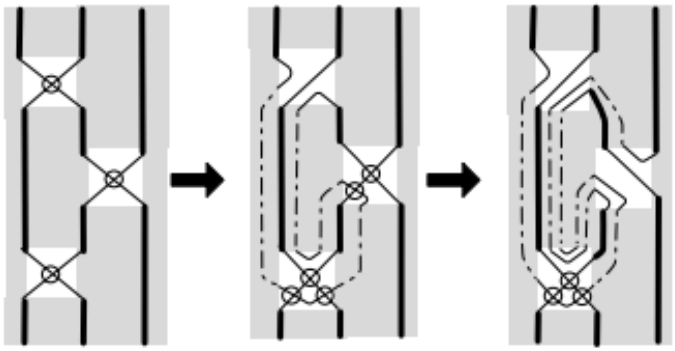
Used: virtual finger, virtual detour

VR2



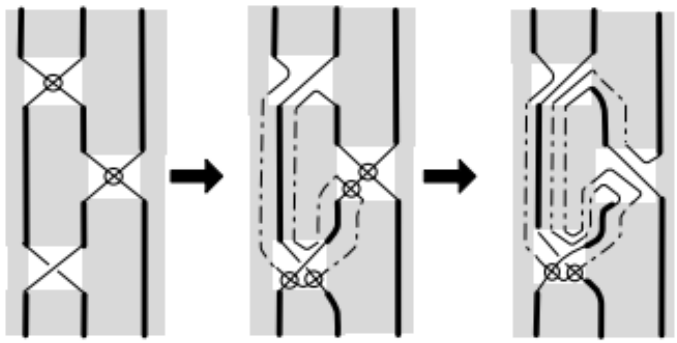
Used: virtual finger, virtual detour

VR3



Used: virtual finger, virtual detour

VR4



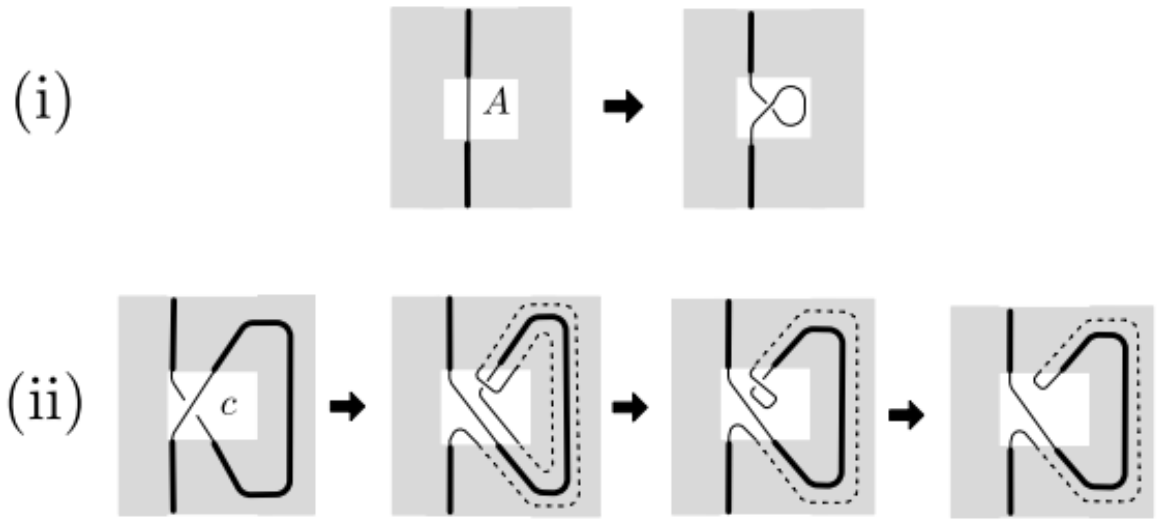
Used: virtual finger, virtual detour

Simple Fact

Observation

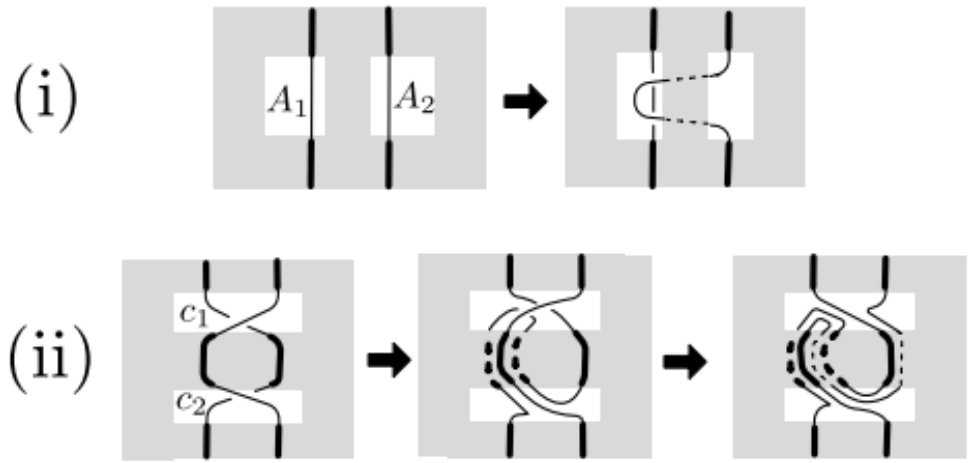
Since performing each of $VR1-4$ does not require WR (and the over detour move), we conclude that if D_1 differs from $D(L_1)$ by only $VR1-4$, then L realizes D_1 as a virtual link diagram.

R1



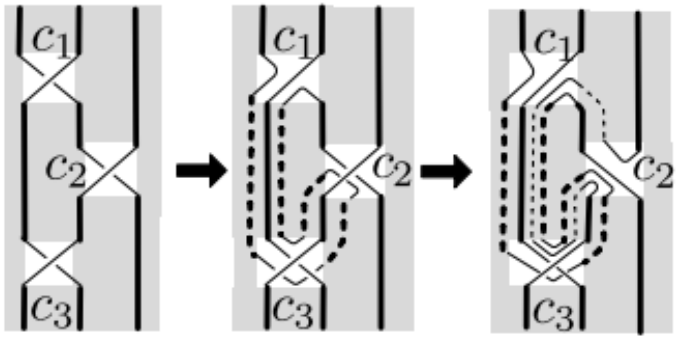
Used: Over finger, over detour

R2



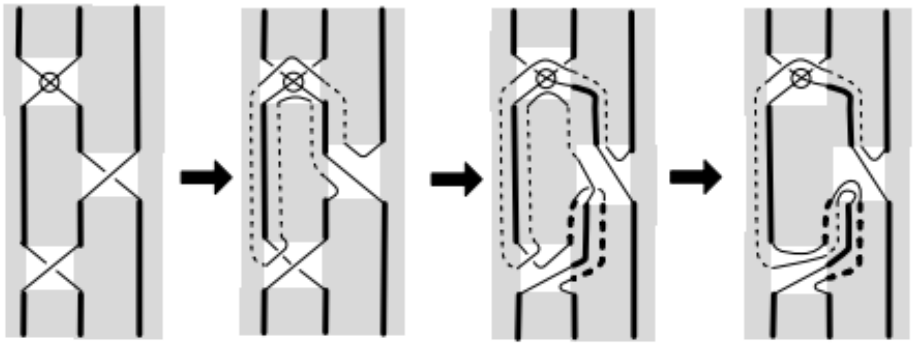
Used: Under finger, over detour

R3



Used: Under finger, over detour

WR



Used: Under finger, over finger, over detour

Simple Fact

Observation

Performing $R1-3$ on an arbitrary diagram requires WR (via the over detour move). Hence, the proof fails for virtual link diagrams.

A Conjecture

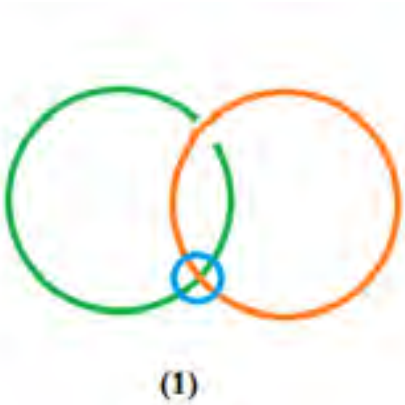
It seems that adding more components will not help a link to realize its subdiagrams.

Conjecture

If a link L has a sublink L_1 which cannot realize all subdiagrams of L_1, then neither can L. The converse trivially does not hold (consider a knot as a sublink).

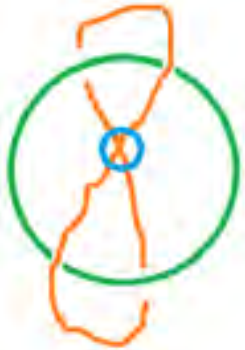
If this conjecture is true, then any suitable invariants that detect whether links can realize all their subdiagrams should satisfy a similar property.

More Examples



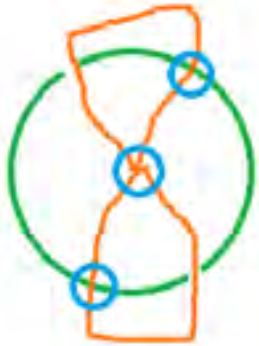
(1)

(2)

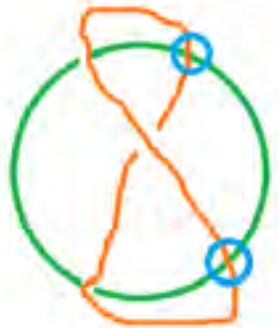


(3)

(4)



(5) Unlink



(6) Kamada's example

Problem

Unlinked links are the only links we have found that can realize all subdiagrams. Even worse, we haven't found a way to prove that any linked link can realize its subdiagrams.

Problem

How to prove a link can realize any subdiagrams?

Reference

Kamada, Naoko. "Diagrams realizing prescribed sublink diagrams for virtual links and welded links." arXiv preprint arXiv:2205.11688 (2022).