Sublinks Realizing Subdiagrams and Forbidden R2 Move

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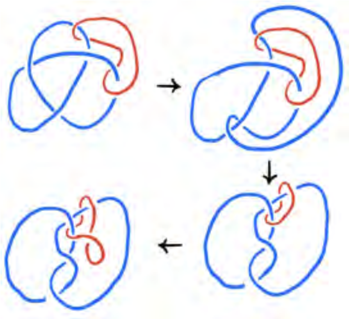
The Problem

Given a virtual link L, partitioned into sublinks $L_1,\ldots,L_n$, and for any possible diagram D_i of L_i, is there a diagram D of L whose subdiagram $D(L_i)$ of L_i is isotopic to D_i for $i = 1, \ldots n$?



Adding Sublinks

Conjecture

Let L be a virtual link partitioned into sublinks $L_1, \dots, L_{n+1}$. Let $L' = L - L_{n+1}$. If L can realize all diagrams of $L_1, ..., L_{n+1}$, then L' can realize all diagrams of $L_1, ..., L_n$.

The contrapositive tells us that no sublink can be added to a link that cannot realize all its subdiagrams such that the resulting link can.

Proof

Let D be a diagram of L, and $D' = D(L')$. Let D_i be a diagram of (L_i) for $i = 1, ..., n$. Since L can realize all its subdiagrams, there is a sequence of generalized Reidemeister moves $M = m_1, ..., m_k$ on D such that when applied to D, the output D_r satisfies that $D_r(L_i)$ is isotopic to D_i for $i = 1, ..., n + 1$.

Proof

We remove all moves acting upon D_{n+1} from the sequence M. We construct a new sequence $M^* = m_1^*, ..., m_k^*$ acting on D', and denote the diagram of L' after m_j by $D'(m_j)$. Let $m_j^* = m_j$ if m_j does not act upon D_{n+1}, and m_j^* be the identity otherwise.

We proceed by checking each m_j.

Proof

Case 1: m_j acts solely on D'. Since D_{n+1} is not involved, $m_j^* = m_j$ and $D'(m_j^*) = D'(m_j)$

Case 2: m_j acts solely on D_{n+1}. Then m_j^* is the identity and since m_j does not effect D', $D'(m_j^*) = D'(m_j)$

Case 3: m_j acts on both D' and D_{n+1}. It is easy to check that each of the the generalized Reidemeister moves that m_j can be don't change D' up to isotopy. So $D'(m_j^*)$ is isotopic to $D'(m_j)$.

Proof

Since each m_j acting upon D_{n+1} only affected D' by isotopy, their replacement with identity moves in M' does not affect the following moves in M'. Hence, when M' is applied to D', the result is isotopic to $D_r - D_r(L_{n+1})$.

Adding Sublinks

Corollary

Let L be a virtual link partitioned in sublinks $L_1 \ldots L_n$. If L can realize all diagrams of $L_1, ..., L_n$, then any sublink L' of L can realize all diagrams of each $L_i \subset L'$.

Proof

Let $L' = \left(\bigsqcup_{i\in\ell} L_i\right) \sqcup L_r$, where ℓ is some index subset of $(1, ..., n)$ and L_r is a disjoint union of sublinks of $L_i \notin L'$ for $i \notin \ell$.There are two cases to consider: $L_r = \emptyset$ and $L_r \neq \emptyset$.

Proof

Case 1: $L_r = \emptyset$. In this case, $L' = \bigsqcup_{i \in \ell} L_i$. Starting with L, remove L_i for $i \notin \ell$ one at a time. At each step, the link still can realize all its subdiagrams by the preceding theorem. Hence, L' can realize all its subdiagrams.

Proof

Case 2: $L_r \neq \emptyset$. In this case, $L' = \left(\bigsqcup_{i \in \ell} L_i\right) \sqcup L_r$. Remove $L_i - L_r$ for $i \notin \ell$ one at a time. At each step, the link can still realize all its subdiagrams by a slight modification of the previous theorem. Hence, L' can realize all its subdiagrams.

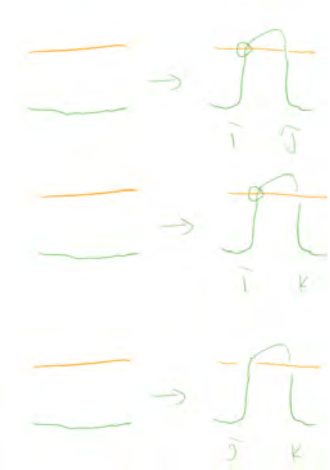
Motivation

We say a link with a partition is good if it can realize all subdiagram and bad otherwise.

By the previous proposition, both good and bad links are built from good sublinks. No sublink is born to make the link bad but the way they are linked makes it. It might be helpful to study how sublinks are linked.

Unlinks are good; If two sublinks intersect each other by virtual crossings, then they can be separated by virtual detours.
How about other links?

Forbidden R2 move

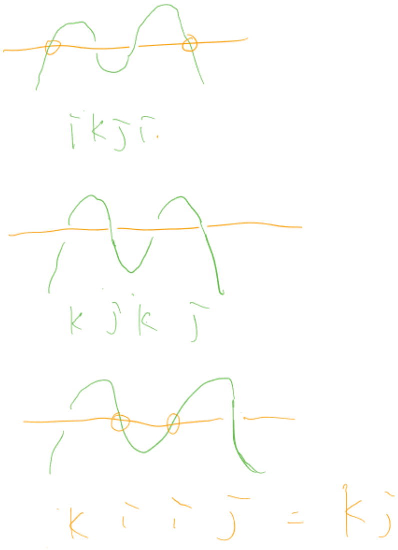


Forbidden R2 move

We denote a forbidden move by the crossings it generates. i for a virtual crossing, j/k for a classical crossing where the shifted arc is over/under the fixed arc.

If multiple forbidden moves happens on the same arcs, they could be simplified by usual R2 and VR2 moves.

Forbidden R2 move



$ik + ji = VR2 + kj$.
Additional linking may kill such avoidance?

Linking

For classical links we have the following theorem:

Theorem

Any classical links can be obtained from unlinks if we allow two arcs pass through each other.

Linking

Do we have a similar theorem for virtual links?

Conjecture

Any virtual links can be obtained from unlinks if we allow "forbidden R2 move".

In particular, we want to explain linking sublinks by forbidden R2 moves:

Conjecture

Given a diagram of L partitioned into sublinks $L_1, \ldots, L_n$, can we always use the forbidden R2 moves to separate the subdiagrams without changing the subdiagrams up to isotopy in the process?

Notations

For convenience, in the following sections, an arc means the piece of a component between two crossing of the same sublink.

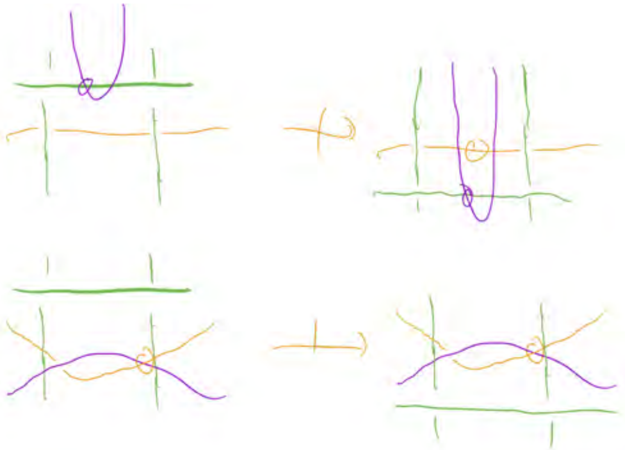
An Arc

Forbidden R2 Move Fixing an Arc

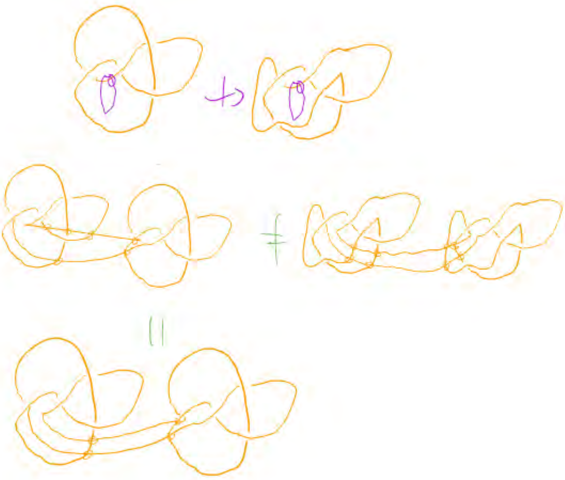
Forbidden R2 move (especially ij, ik types) characterizes some bad links.

In many simple links, if an unavoidable ij or ik move is applied on two arcs, then the arcs can neither completely over-detour anything nor be completely over-detoured by. So they are "fixed". With the method we can immediately creates many examples.

Forbidden R2 Move Fixing an Arc



Forbidden R2 Move Fixing an Arc

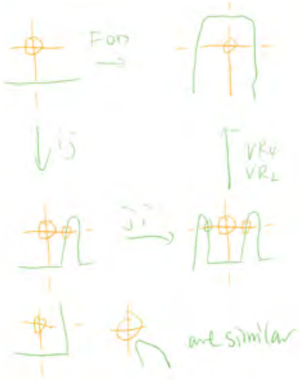


Forbidden R2 Move Fixing an Arc

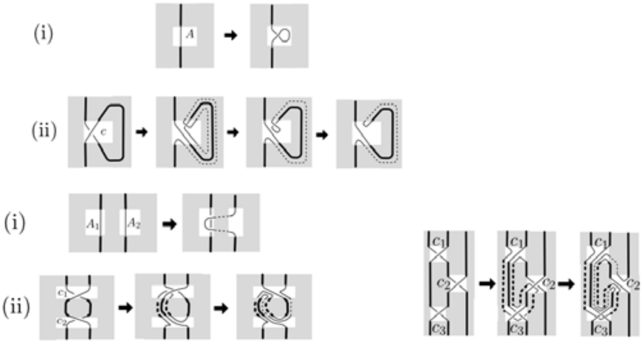
Forbidden R2 Move $\implies$ WR Move

Any Welded move can be done by forbidden R2 move.

Forbidden R2 move is stronger than welded move.



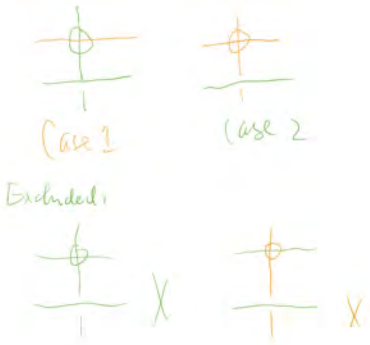
What Kamada's Method Tells



Recall that local R1-3 moves are the problematic cases for general virtual links.
R2 (i) can be fixed by replacing over detour by virtual detour and R2.

Forbidden R2 Move $\implies$ WR Move

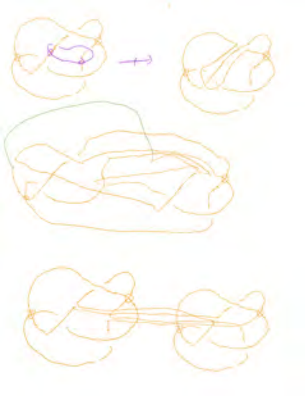
During those over detour moves, an arc does not over-detour any virtual crossing from the same sublink or create new crossings on the sane sublink. So the problematic over detour moves are the following cases:



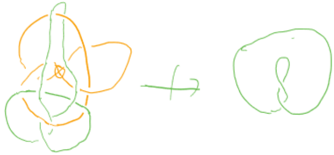
Forbidden R2 Move $\implies$ WR Move

Case 1 and 2 can be seen as consequence of *ij* and *kj* moves applied **on two sublinks**, i.e., determined by how sublinks are linked. We intentionally avoid any forbidden R2 moves performed on one sublink, otherwise new arcs will be created.

Forbidden R2 Move $\implies$ WR Move



Case 1

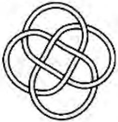


(Possible) Case 2

Symmetry

If a link is symmetric, then fixing one arc may not stop we realizing a diagram by the symmetric arc. The simplest link diagram——unknot doesn't even have an arc by our definition. So we avoid these special cases for the moment.





Conjecture

Conjecture

If sublink L_1 is not symmetric and the process of separating L_1 from the rest sublinks involves unavoidable ij, ik types of forbidden R2 move, then there exists a diagram of L_1 that is unrealizable.

Problem

"Unavoidable" to be well defined.

To Be Continued...