Unknot Using Forbidden Moves and Bounds on Realizable Partition Size

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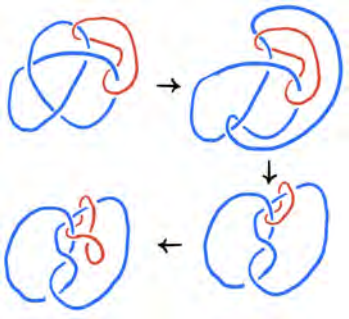
The Problem

Given a virtual link L, partitioned into sublinks $L_1,\ldots,L_n$, and for any possible diagram D_i of L_i, is there a diagram D of L whose subdiagram $D(L_i)$ of L_i is isotopic to D_i for $i = 1, \ldots n$?



Conjectures

In the last presentation we introduced forbidden R2 moves in attempts to describe how the sublinks are linked.

Conjecture 1

Every virtual link can be unknotted if we allow forbidden R2 moves.

Conjecture 2

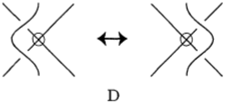
Every sublink of a virtual link can be separated by forbidden R2 move.

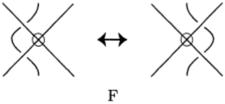
Unknot a Virtual Knot

A virtual knot can be unknotted by two moves:

Theorem 1 (Taizo Kanenobu)

For any virtual knot K, there exists a finite sequence of generalized Reidemeister moves and D- and F-moves that takes K to a trivial knot.

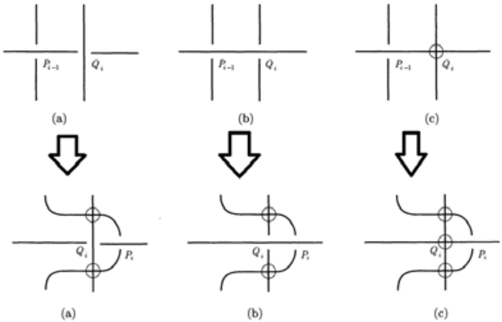




Unknot a Virtual Knot

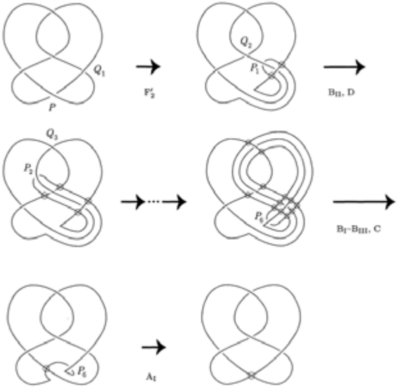
The idea of proof is to replace a classical crossing by a virtual crossing by performing D and F move. After every crossing becomes virtual, it is an unknot.

The following are derived from D and F moves and will compose finger moves.



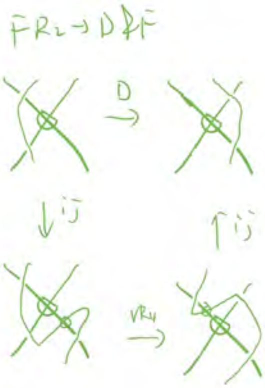
Unknot a Virtual Knot

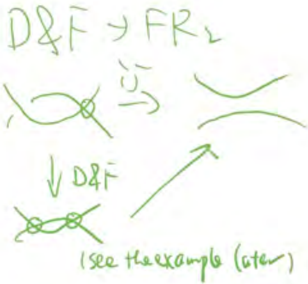
Then we can use the mentioned moves to create "virtual skin finger move" along the arc.



Unknot a Virtual Knot

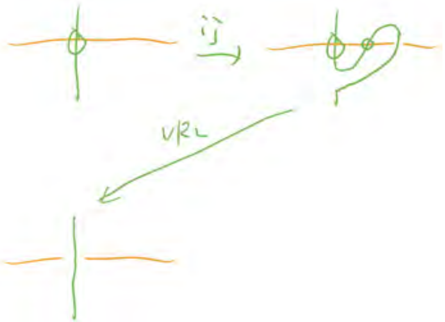
For virtual knot, forbidden R2 moves are equivalent to D and F moves so Theorem 1 is equivalent to Conjecture 1 in knot case.





Unknot a Virtual Knot

In fact forbidden R2 moves is a bad generalization to the D and F moves because it is just equivalent to changing the type of a crossing.



Updated Conjecture

But if we restrict it to where two sublinks cross, then we may have more useful information.

Conjecture 3

If the minimal number of virtual crossings between two (not both unknot) sublinks is greater than 0, then the link has a unrealizable subdiagram.

Motivating Question

All links can be partitioned into k sublinks in such a way that they can realize all their subdiagrams. For classical and welded links, the maximum k is equal to the number of components in the link. But, what is the maximum k for virtual (and twisted) links?

Bounds on Maximal Realizable Partition Cardinality

Theorem

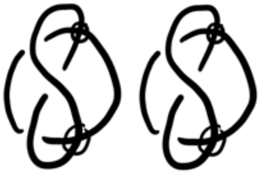
For any (classical, welded, virtual, or twisted) link L,

$$u_L \leq k_L \leq n - \left\lceil \frac{m}{2} \right\rceil .$$

Where u_L is the number of unlinked sublinks of L, k_L is the maximum cardinality of a realizable partition of L, and n is the number of components of L, and m is the number of partitioned unrealizable sublinks in a sublink of L.

Proof (Lower Bound)

An obvious lower bound for the maximum k is the number of unlinked sublinks of L. Such sublinks can realize all their diagrams by definition.



Proof (Upper Bound)

An obvious upper bound is the number of components of L, which we call n. But how can we refine this bound? If we know that L contains an unrealizable sublink, we can use this knowledge to refine the bound.

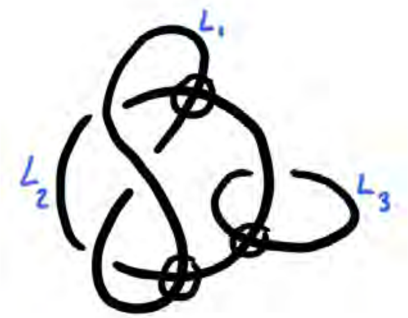
Proof (Upper Bound)

Let L' be a sublink of L partitioned into $L'_1, ..., L'_m, ..., L'_v$ such that L' cannot realize the diagrams of the L'_i for $i = 1, ..., m$ and cannot realize all their sublinks. When repartitioning L so that it can realize all its subdiagrams, each of the L'_i for $i = 1, .., m$ cannot be split into sublinks because their constituents are unrealizable. Hence, each of the L'_i for $i = 1, ..., m$ must be either paired with at least another unrealizable sublink or one other component of the diagram. Hence the upper bound may be refined to $n - \left\lceil \frac{m}{2} \right\rceil$.

Example

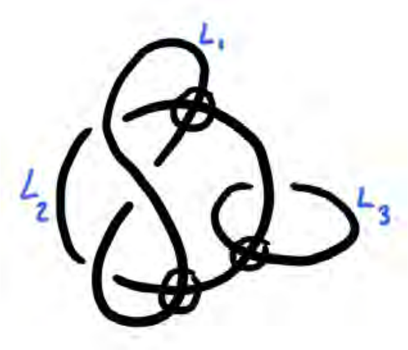
We know that the sublink $L' = L_1 \sqcup L_2$ is unrealizable when partitioned as such. Of L_1 and L_2, we know that at least L_1 cannot realize all its diagrams. Hence we know that

$$u_L \leq k_L \leq n - \left\lceil \frac{m}{2} \right\rceil$$
$$1 \leq k_L \leq 3 - 1 = 2.$$



Example (Cont'd)

One possibly realizable partition of L is $L = L_1^* \sqcup L_2^*$, where $L_1^* = L_1 \sqcup L_2$ and $L_2^* = L_3$. But is this partition actually realizable?



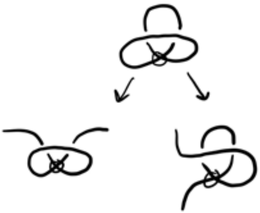
Final Thoughts

Conjecture

Unknots without self crossings are realizable, no matter how they are linked.

The thought is to consider long knots, or knots whose ends go to infinity rather than making a closed loop. For classical knots, closed knots and long knots are isomorphic. But this is not the case for virtual knots. Specifically, there exist virtual unknots that can be made nontrivial long knots by cutting at certain points.

Final Thoughts



If there exists a way to cut any virtual unknot such that it is equivalent to the long unknot, then the conjecture is true. This follows by moving all crossings on the non-self-crossing-unknot to one small disc, stretching the knot out arbitrarily, and treating is a long unknot.