Definitions of matroids

References: J. Oxley [Ox], D. J. A. Welsh [Wel], H. Whitney [Wh].

Independent sets.

A matroid is a pair $M = (E, \mathcal{I})$ consisting of a finite set E and a nonempty collection $\mathcal{I}$ of its subsets, called *independent sets*, satisfying the axioms:

(**I1**) *Any subset on an independent set is independent.*

(**I2**) *If X and Y are independent and $|X| = |Y| + 1$, then there is an element $x \in X - Y$ such that $Y \cup x$ is independent.*

Circuits.

A matroid is a pair $M = (E, \mathcal{C})$ consisting of a finite set E and a nonempty collection $\mathcal{C}$ of its subsets, called *circuits*, satisfying the axioms:

(**C1**) *No proper subset of a circuit is a circuit.*

(**C2**) *If C_1 and C_2 are distinct circuits and $c \in C_1 \cap C_2$, then $(C_1 \cup C_2) - c$ contains a circuit.*

Bases.

A matroid is a pair $M = (E, \mathcal{B})$ consisting of a finite set E and a nonempty collection $\mathcal{B}$ of its subsets, called *bases*, satisfying the axioms:

(**B1**) *No proper subset of a base is a base.*

(**B2**) *If B_1 and B_2 are bases and $b_1 \in B_1 - B_2$, then there is an element $b_2 \in B_2 - B_1$ such that $(B_1 - b_1) \cup b_2$ is a base.*

Rank function.

A matroid is a pair $M = (E, r)$ consisting of a finite set E and a function r, *rank*, assigning a number to a subset of E and satisfying the axioms:

(**R1**) *The rank of an empty subset is zero.*

(**R2**) *For any subset X and any element $y \notin X$,*

$$r(X \cup \{y\}) = \begin{cases} r(X)\ , & \textit{or} \\ r(X) + 1\ . \end{cases}$$

(**R3**) *For any subset X and two elements y,z not in X, if $r(X \cup y) = r(X \cup z) = r(X)$, then $r(X \cup \{y, z\}) = r(X)$.*

Properties.

- For an independent set X, the rank is equal to its cardinality, $r(X) = |X|$.
- Circuits are minimal dependent subsets.
- Circuits are the subsets X with $r(X) = |X| - 1$.
- A base is a maximal independent set.
- Rank of a subset X is equal to the cardinality of the maximal independent subset of X.
- All bases have the same cardinality which is called the *rank of matroid*, $r(M)$.
- Rank of a subset X is equal to the cardinality of the maximal independent subset of X.

Examples.

1. The ***cycle matroid*** $\mathcal{C}(G)$ of a graph G. The underlying set E is the set of edges $E(G)$. A subset $X \subset E$ is independent if and only if it does not contain any cycle of G. A base consist of edges of a spanning forest of G. The rank function is given by $r(X) := v(G) - k(X)$, where $v(G)$ is the number of vertices of G and $k(X)$ is the number of connected components of the spanning subgraph of G consisting of all the vertices of G and edges of X.

2. The ***bond matroid*** $\mathcal{B}(G)$ of a graph G. The circuits of $\mathcal{B}(G)$ are the minimal edge cuts, also known as the *bonds* of G. These are minimal collections of the edges of G which, when removed from G, increase the number of connected components. The rank $r(X)$ is equal to the maximal number of edges deletion of which do not increase the number of connected components of the spanning subgraph wich edges from X.

3. The ***uniform matroid*** $U_{k,n}$ is a matroid on an n-element set E where all subsets of cardinality $\leqslant k$ are independent. For the complete graph K_3 with three vertices, $\mathcal{C}(K_3) = U_{2,3}$. The matroid $U_{2,4}$ is not *graphical*. That is there is no any graph G such that $\mathcal{C}(G) = U_{2,4}$. It is also not *cographical*. That is there is no any graph G such that $\mathcal{B}(G) = U_{2,4}$.

4. A finite set of vectors in a vector space over a filed $\mathbb{F}$ has a natural matroid structure which is called ***representable*** (over $\mathbb{F}$). We may think about the vectors as column vectors of a matrix. The rank function is the dimension of the subspace spanned by the subset of vectors, or the rank of the corresponding submatrix. The cycle matroid $\mathcal{C}(G)$ is representable (over $\mathbb{F}_2$). The correspondent matrix is the incidence matrix of G, i.e. the matrix whose (i,j)-th entry is 1 if and only if the i-th vertex is incident to the j-th edge. The uniform matroid $U_{2,4}$ is not representable over $\mathbb{F}_2$, but it is representable over $\mathbb{F}_3$.

Dual matroids. Given any matroid M, there is a dual matroid M^* with the same underlying set and with the rank function given by $r_{M^*}(H) := |H| + r_M(M \setminus H) - r(M)$. In particular $r(M) + r(M^*) = |M|$. Any base of M^* is a complement to a base of M. The bond matroid of a graph G is dual to the cycle matroid of G: $\mathcal{B}(G) := (\mathcal{C}(G))^*$.

Dual representable matroids. Let $E = \{v_1, \ldots, v_n\}$ be a collection of vectors in a vector space U and M be a matroid of their linear dependences. Consider an n-dimensional vector space V with a basis $e_1, \ldots, e_n$ and a linear map $f : V \to U$ sending e_k to v_k. Denote the kernel of this map by W. It is a subspace of V and there is a natural inclusion map $i : W \hookrightarrow V$. There is the dual map $W^* \xleftarrow{i^*} V^*$ of dual vector spaces. The space V^* has a natural dual basis $e_1^*, \ldots, e_n^*$. Their images $i^*(e_1^*), \ldots, i^*(e_n^*)$ is a collection of vectors in the space W^*. These vectors with the structure of linear dependences between them form the dual matroid M^*.

The Whitney planarity criteria [Wh]**.** A graph G is planar if and only if its bond matroid $\mathcal{B}(G)$ is graphical. In this case, it will be the cycle matroid of the dual graph, $\mathcal{B}(G) = (\mathcal{C}(G))^* = \mathcal{C}(G^*)$.

Tutte polynomial.
$$\boxed{T_M(x,y) \;:=\; \sum_{X \subseteq E} (x-1)^{r(E)-r(X)}(y-1)^{n(X)}}$$

References

[Ox] J. Oxley, *What is a matroid?*, preprint `http://www.math.lsu.edu/ oxley/survey4.pdf`.

[Wel] D. J. A. Welsh, *Matroid Theory*, Academic Press, London, New York, 1976.

[Wh] H. Whitney, *On the abstract properties of linear dpendence*, Amer. J. Math. **57**(3) (1935) 509–533.