Knots and Graphs

Working Group [Summer 2022]

MATH 4193, class number 16664

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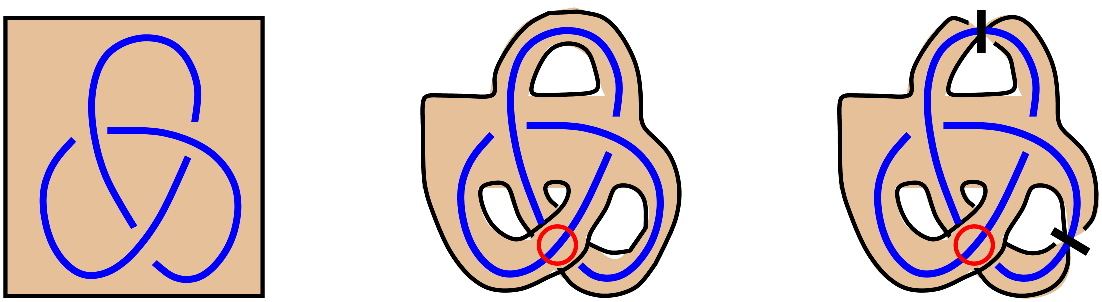
RESEARCH PROJECTS

Project 1. ***Twisted knots.*** (Kabir Belgikar, Calvin Forsee, Wo Wu)

Twisted knot theory was introduced by M.O.Bourgoin [Bou] as a generalization of virtual knot theory by L. Kauffman in [Ka]. One can think about virtual knots as diagrams on (orientable) surfaces and about twisted knots as diagrams on a non-orientable surfaces.



`knot` `virtual knot` `twisted knot`

We plan to start this project with studying the paper [XD] about the forbidden moves which would allow to untie a twisted knot to the unknot. The goal is to try to settle Conjecture 3.1 of this paper which claims that the minimal cardinality of a set of forbidden moves is 3.

Project 2. ***Subdiagrams of virtual links.*** (Mark Kikta, Yan Xuan)

Welded link theory is another generalization of virtual link theory when one of the two forbidden moves is allowed.

A well known theorem of G.T.Jin and J.H.Lee states that the give diagrams of sublinks can be realized in the entire diagram of a link with these sublinks. Recently this theorem was generalized to welded links by N.Kamada [Kam]. Also it was shown that this theorem fails for general virtual links. The problem is how to describe virtual diagrams for which the theorem does hold. Is there an extension of this theorem to twisted links?

Project 3. ***Delta-matroids.*** (Leon Lozinskiy, Daniel Yuschak)

A *delta-matroid* is a fine set E with a collection of its subsets $\mathcal{F}$, calling *feasible sets*, satysfyling the *Symmetric Exchange Axiom*: $\forall X, Y \in \mathcal{F}$ and $\forall u \in X\Delta Y, \exists v \in X\Delta Y$ such that $X\Delta\{u,v\} \in \mathcal{F}$, where $X\Delta Y := (X \cup Y) \setminus (X \cap Y)$ is the symmetric difference of sets.

About delta-matroid see [Mof]. A reach source of delta matrids is given by ribbon graphs, where the set E is the set of edges of a ribbon graph and a subset of edges is feasible if it forms a quasi-tree (that is the corresponding spanning subgraph has a single boundary component).

For a delta-matroid $D = (E, \mathcal{F})$ and $A \subseteq E$ one can define the *twist* of D by A, $D * A := (E, \mathcal{F}')$, where $\mathcal{F}' := \{X \Delta A : X \in \mathcal{F}\}$. In case of ribbon graphs the twist operation has a nice geometrical interpretation called *partial duality*.

The width $w(D)$ of a delta-matroid D is the difference between the sizes of its largest and smallest feasible sets. The *twist polynomial* of a delta-matroid $D = (E, \mathcal{F})$ is the generating function

$$^{\partial}w_D(z) := \sum_{A \subseteq E} z^{w(D*A)}$$

In the case of ribbon graphs the twist polynomial is called *partial-dual genus polynomial*. In a recent paper [ChVT] all ribbon graphs with one term partial-dual genus polynomial were described. One of the problem formulated in this paper was about generalization of this description to the twist polynomials of general delta-matroids. Partially, for *binary* delta-matroids, it was solved in a paper [YJ]. The project is aimed to solve it in general. We plan to start looking for one term twist polynomial of delta-matroids with small number of elements, say $|E| = 3, 4$, and try to find some patterns there.

References

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