Heap Colorings

Charlton Li

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Region Coloring

Question: How can we color the regions of a knot diagram by a "quandle-like" structure?

Answer: We can find motivation from quandles and presentations of the knot group.

The Knot Group

Definition 1

The **knot group** of a knot K is defined to be $\pi_1(\mathbb{R}^3 \setminus K)$, i.e., the group of equivalence classes of loops at a fixed basepoint under homotopy (continuous deformation) with concatenation as the operation. Since $\mathbb{R}^3 \setminus K$ is path-connected, the choice of basepoint does not matter.

Group Presentation

Definition 2

The set of **group words** $W_G(X)$ on a set X is defined recursively as follows:

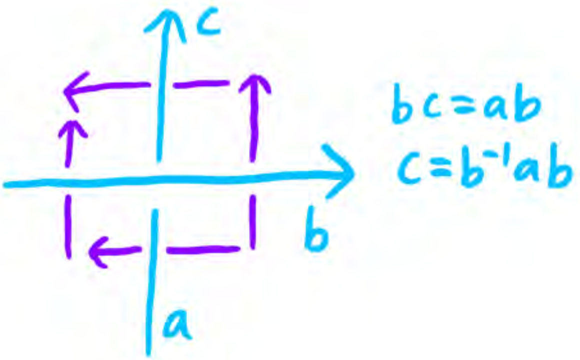
1. $x \in X \implies x, x^{-1} \in W_G(X)$
2. $x, y \in W_G(X) \implies xy \in W_G(X)$

The **free group** on X is $W_G(X)/\sim$, where $\sim$ is the group axioms. A **group presentation** $\langle X|R \rangle$ is then the free group on X modulo the relations R, where R is comprised of pairs (u, v), where $u, v \in W_G(X)$, and are interpreted as $u = v$. Elements of X are called generators.

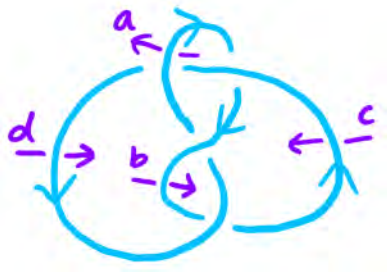
Wirtinger Presentation

Definition 3

The **Wirtinger presentation** of the knot group is obtained by taking the arcs of an oriented knot diagram as generators subject to the relations coming from each crossing:



Example



The Wirtinger presentation for the figure 8 knot is

$$G = \langle a, b, c, d \mid d = a^{-1}ca, b = c^{-1}ac, a = bdb^{-1}, c = dbd^{-1} \rangle$$

We thus obtain a presentation of the **fundamental quandle**:

$$Q = \langle a, b, c, d \mid d = c * a, b = a * c, a = d \,\bar{*}\, b, c = b \,\bar{*}\, d \rangle$$

Quandle Presentation

Definition 4

The set of **quandle words** $W_Q(X)$ on a set X is defined recursively as follows:

1. $x \in X \implies x \in W_Q(X)$
2. $x, y \in W_Q(X) \implies x * y, x \bar{*} y \in W_Q(X)$

The **free quandle** on X is $W_Q(X)/\sim$, where $\sim$ is the quandle axioms:

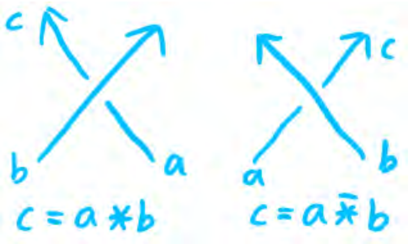
1. $x * x = x$
2. $(x * y) \bar{*} y = x = (x \bar{*} y) * y$
3. $(x * y) * z = (x * z) * (y * z)$

The **quandle presentation** $\langle X | R \rangle$ is then the free quandle on X modulo the relations R, where R is comprised of pairs (u, v), where $u, v \in W_Q(X)$, and are interpreted as $u = v$.

Fundamental Quandle

Definition 5

The **fundamental quandle** of a knot diagram is given by the quandle presentation obtained from taking the arcs to be the generators and imposing relations from each crossing:



A coloring of a knot diagram K by a quandle Q is precisely a homomorphism from the fundamental quandle of K to Q.

Fundamental Quandle

Theorem 6

The fundamental quandle is a complete knot invariant up to orientation and mirror image.

Note that the knot group is not a complete invariant: for instance, the square and granny knots below have isomorphic fundamental groups.

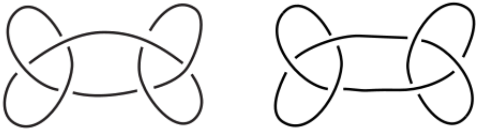
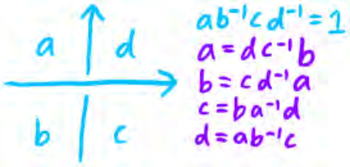


Image credit: Wolfram Mathworld

Dehn Presentation

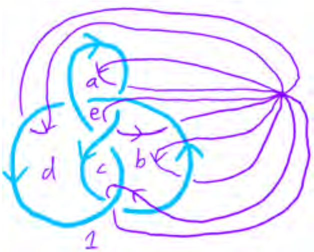
Definition 7

The **Dehn Presentation** of the knot group is obtained by taking the regions of an oriented knot diagram as generators subject to the relations coming from each crossing:



Starting from the outgoing understrand, write down the regions in anticlockwise-manner with alternating exponent. Any one of the regions is then set to 1 (as a convention, we will always choose the unbounded region). Each region corresponds to the element of the knot group that enters through that region and exits out of the region labelled 1.

Example



The Dehn presentation for the figure 8 knot is

$$G = \langle a, b, c, d, e \mid e = b1^{-1}a, e = a1^{-1}d, c = be^{-1}d, c = d1^{-1}b \rangle$$

Similarly to how the relations of the Wirtinger presentation can be expressed using only conjugation, the relations of the Dehn presentation can be expressed using only the ternary operation $(x, y, z) \mapsto xy^{-1}z$.

Heaps

Definition 8

The group G as a set with the operation $(x, y, z) \mapsto xy^{-1}z$ is called the **group heap** of G, denoted $\mathrm{Prin}(G)$.

Definition 9

A **heap** is a set X with a ternary operation $(a, b, c) \mapsto [a, b, c]$ satisfying

1. Para-associativity: For all $x_1, x_2, x_3, x_4, x_5 \in X$,

$$\big[[x_1, x_2, x_3], x_4, x_5\big] = \big[x_1, x_2, [x_3, x_4, x_5]\big]$$

2. Mal'cev identities: For all $x, y \in X$,

$$[x, y, y] = x = [y, y, x]$$

Relation Between Groups and Heaps

Proposition 1

The group heap $\mathrm{Prin}(G)$ *of a group* G *is indeed a heap.*

Conversely, if H *is a heap with a distinguished element* $e \in H$*, then the operation* $(a, b) \mapsto [a, e, b]$ *gives* H *a group structure where* e *is the identity and* $x^{-1} = [e, x, e]$ *for any* $x \in H$*. This group is denoted* $\mathrm{Aut}(H; e)$*.*

Furthermore, these two transformations are mutual inverses, although one direction is only up to isomorphism:

$$\mathrm{Prin}(\mathrm{Aut}(H; e)) = H, \qquad \mathrm{Aut}(\mathrm{Prin}(G); e) \cong G$$

A heap is an "affine" version of a group, i.e. a group after one forgets the identity.

Another Associativity Property of Heaps

The following is another associativity property for heaps that is commonly listed as an axiom, but it can be proved from the other axioms.

Proposition 2

Let H be a heap. Then for all $x_1, x_2, x_3, x_4, x_5 \in H$,

$$\big[[x_1, x_2, x_3], x_4, x_5\big] = \big[x_1, [x_4, x_3, x_2], x_5\big]$$

Proof:

$$\begin{aligned}\big[x_1, [x_4, x_3, x_2], x_5\big] &= x_1(x_4x_3^{-1}x_2)^{-1}x_5 \\ &= x_1x_2^{-1}x_3x_4^{-1}x_5 \\ &= (x_1x_2^{-1}x_3)x_4^{-1}x_5 \\ &= \big[[x_1, x_2, x_3], x_4, x_5\big]\end{aligned}$$

Heap Presentation

Definition 10

The set of **heap words** $W_H(X)$ on a set X is defined recursively as follows:

1. $x \in X \implies x \in W_H(X)$
2. $x, y, z \in W_H(X) \implies [x, y, z] \in W_H(X)$

The **free heap** on X is $W_H(X)/\sim$, where $\sim$ is the heap axioms:

1. $\big[[x_1, x_2, x_3], x_4, x_5\big] = \big[x_1, x_2, [x_3, x_4, x_5]\big]$
2. $[x, y, y] = x = [y, y, x]$

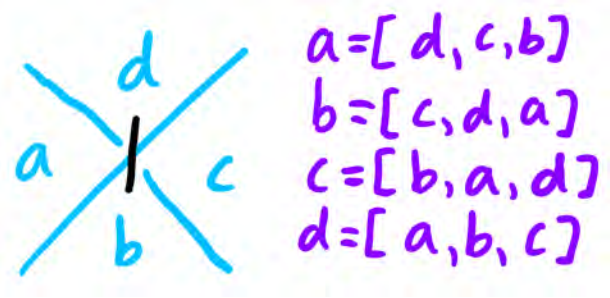
The **heap presentation** $\langle X | R \rangle$ is then the free heap on X modulo the relations R, where R is comprised of pairs (u, v), where $u, v \in W_H(X)$, and are interpreted as $u = v$.

Positive Marker

To define the fundamental heap (or Dehn presentation) of a knot, it is not necessary to have an orientation: using the positive marker is enough.

Fundamental Heap

We can associate a heap to a given oriented knot diagram by taking the regions to be the generators and getting a relation from each crossing.



The variables for the heap operation are taken in anticlockwise order if the region contains the positive marker, and in clockwise order if it doesn't.

Fundamental Heap

Theorem 11

The fundamental heap is a knot (and link) invariant.

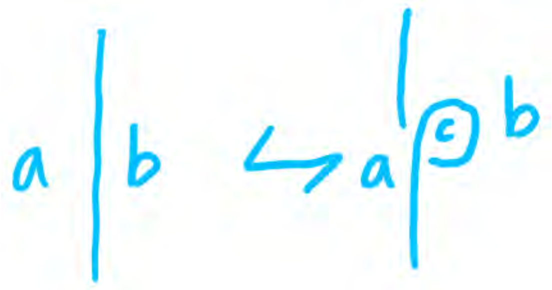
To prove this, we need the following ways to transform a presentation:

Lemma 12

Tietze operations for heaps:
Let $\langle X|R\rangle$ be a finite presentation of a heap H (so X and R are both finite). Then any finite presentation of H can be obtained from $\langle X|R\rangle$ by the following operations and their inverses:

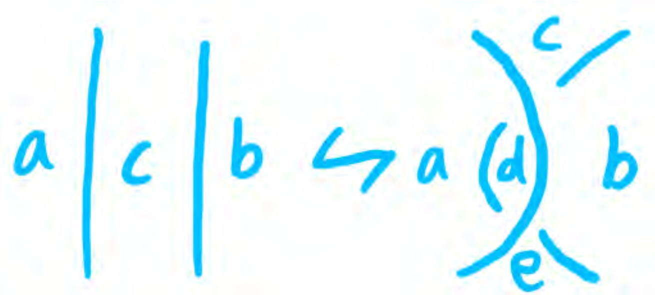
- T1 *If (u, v) can be derived from R, then replace R by $R \cup \{(u, v)\}$*
- T2 *If $u \in W_H(X)$ and y is a letter not occurring in X, then replace X by $X \cup \{y\}$ and R by R by $R \cup \{(y, u)\}$*

Invariance Under R1



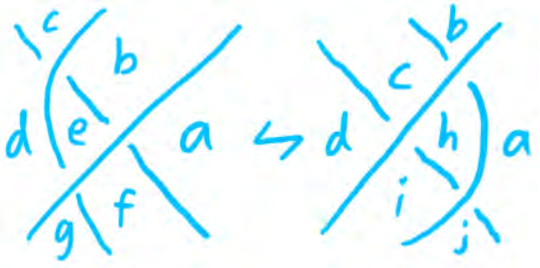
This is just (T2) with $c = [b, a, b]$.

Invariance Under R2



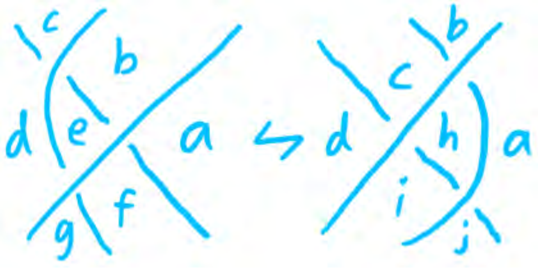
The relations $d = [a, c, b]$ and $e = [b, d, a]$ are added. They imply $e = [b, [a, c, b], a] = \big[[b, b, c], a, a\big] = c$, which we can add to R by (T1). Then $e = [b, d, c]$ is a consequence of $e = c$ and $d = [a, c, b]$, so it can be removed with $(\mathrm{T1})^{-1}$. Replace e by c in all relations except $e = c$, then remove e and $e = c$, along with d and $d = [a, c, b]$ using $(\mathrm{T2})^{-1}$ to obtain the original presentation.

Invariance Under R3



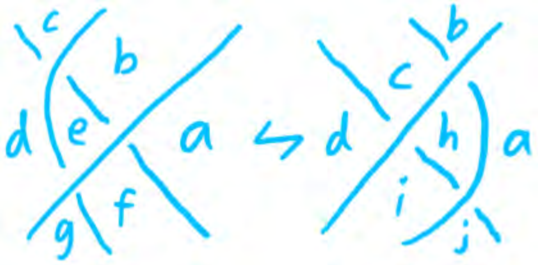
The presentation on the left has the relations (i) $e = [b, c, d]$, (ii) $f = [a, b, e]$, (iii) $g = [f, e, d]$. Add the consequences (iv) $f = [a, b, [b, c, d]]$ and (v) $g = [[a, b, [b, c, d]], [b, c, d], d]$ using (T1) and remove (ii) and (iii) as they are now consequences of (i),(iv),(v). Remove e and (i) using $(T2)^{-1}$. Make the substitutions (iv) and (v) in all other relations, then remove f,g,(iv),(v) by $(T2)^{-1}$. Call this presentation P_1.

Invariance Under R3



The presentation on the right has the relations (i') $h = [a, b, c]$, (ii') $i = [h, c, d]$, (iii') $j = [a, h, i]$. Add the consequences (iv') $i = [[a, b, c], c, d]$ and (v') $j = [a, [a, b, c], [[a, b, c], c, d]]$ using (T1) and remove (ii') and (iii') as they are now consequences of (i'), (iv'), (v'). Remove h and (i') using $(T2)^{-1}$. Make the substitutions (iv') and (v') in all other relations, then remove i, j, (iv'), (v') by $(T2)^{-1}$. Call this presentation P_2.

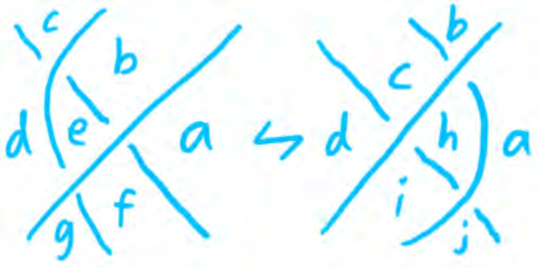
Invariance Under R3



The equivalence of P_1 and P_2 follows from the fact that

$$\begin{aligned} j &= [a, [a, b, c], [[a, b, c], c, d]] \\ &= [a, [a, b, c], [a, b, d]] \\ &= [a, c, [b, a, [a, b, d]]] \\ &= [a, c, d] \\ &= f \end{aligned}$$

Invariance Under R3



and

$$
\begin{aligned}
g &= [[a, b, [b, c, d]], [b, c, d], d] \\
&= [[a, c, d], [b, c, d], d] \\
&= [[[a, c, d], d, c], b, d] \\
&= [a, b, d] \\
&= i
\end{aligned}
$$

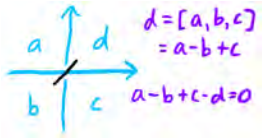
Heap Coloring

Definition 13

A coloring of a knot diagram K by a heap H is a homomorphism from the fundamental heap of K to H.

Proposition 3

Let $H = \text{Prin}(\mathbb{Z}/n\mathbb{Z})$*. Then coloring by* H *gives a modified Dehn n-coloring:*



Working with the operation $(a, b, c) \mapsto ac^{-1}b$ in a group gives another example of a fundamental ternary algebra that is a knot invariant, and reduces to Dehn n-coloring.

References

- Maciej Niebrzydowski, *On some ternary operations in knot theory* (2013)
- David Joyce, *An Algebraic Approach to Symmetry with Applications to Knot Theory* (1979)
- Tomasz Brezezinski and Bernard Rybolowicz, *Modules over Trusses vs Modules over Rings: Direct Sums and Free Modules* (2019)