Goeritz Matrices, Tait Graphs, Matroids, and Polynomials

Mark Kikta

Goeritz Matrix

Definition

Let $D \subset \mathbb{R}^2$ be a link diagram, and shade one of the checkerboard surfaces bounded by D. Label the unshaded regions of $\mathbb{R}^2 \setminus D$ by $X_0, X_1, \ldots, X_n$. An unreduced Goeritz Matrix $\tilde{G}$ of D is given by

$$\tilde{g}_{ij} = \begin{cases} \sum_{c \text{ adjacent to } X_i \text{ and } X_k} \sigma(c), & i \neq j \\ -\sum_{k \neq i} g_{ik}, & i = j. \end{cases}$$

The **Goeritz matrix** G is obtained by deleting a row and column of $\tilde{G}$.

A Few Matrices

Fix g_{ij} with $i \neq j$.

- Let G'_{ij} be the symmetric matrix obtained from G by the following operations:

$$g_{ii} \mapsto g_{ii} + g_{ij}$$
$$g_{jj} \mapsto g_{jj} + g_{ij}$$
$$g_{ij}, g_{ji} \mapsto 0.$$

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- Let G''_{ij} be the symmetric matrix obtained from G by the following operations:

$$\begin{aligned} &g_{ii} \mapsto g_{ii} + g_{jj} + 2g_{ij} \\ &g_{ik} \mapsto g_{ik} + g_{jk}, \text{ for all } k \neq i \\ &g_{ki} \mapsto g_{ki} + g_{kj}, \text{ for all } k \neq i \\ &\text{Delete the } j\text{th row and column.} \end{aligned}$$

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- Let G'_i be the symmetric matrix obtained by deleting the ith row and column of G.

μ

Definition

Define μ : {symmetric integer polynomials} $\to \mathbb{Z}[A^{\pm 1}]$ recursively:

1. If G is empty, $\mu[G] = 1$.
2. For any g_{ij} with $i \neq j$

$$\mu[G] = A^{-g_{ij}} \mu[G'_{ij}] + P_{-g_{ij}}(A) \mu[G"_{ij}].$$

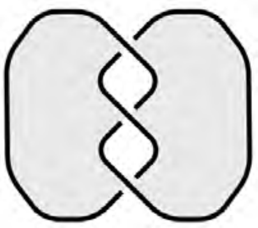
3. If g_{ii} is such that $g_{i\ell} = 0$ for all $\ell \neq i$, then

$$\mu[G] = (A^{g_{ii}}(-A^{-2} - A^2) + P_{g_{ii}}(A)) \mu[G'_i].$$

Where $P_n \in \mathbb{Z}[A^{\pm 1}]$ with $n \in \mathbb{Z}$ is defined by

$$P_n(A) = \sum^{|n|} (-1)^{j-1} A^{\text{sgn}(n)(|n|-4j+2)}.$$

Example



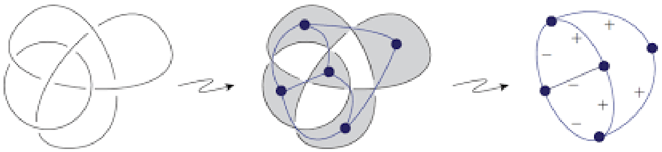
$$\tilde{G} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, G = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

$$\begin{aligned} \mu \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} &= A\mu[G'_{12}] + P_1(A)\mu[G''_{12}] \\ &= A\mu \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + A^{-1}\mu[2] \\ &= A(A(-A^{-2} - A^2) + P_1(A))\mu[1] \\ &+ A^{-1}(A^2(-A^{-2} - A^2) + P_{-2}(A))\mu[] \\ &= -A^4\mu[1] + (-A^3 - A^{-5}) \\ &= -A^4(A(-A^{-2} - A^2) + P_1(A)) - A^3 - A^{-5} \\ &= A^7 - A^3 - A^{-5}. \end{aligned}$$

Tait Graphs

Definition

Let D be a checkerboard-colorable link diagram, and let S be a checkerboard surface bounded by D. The **Tait graph** of D and S is the signed graph formed by assigning a vertex to each region of S and an edge to each crossing $c \in D$ that connects the vertices assigned to shaded regions adjacent to c. Assign the weight $\sigma(c)$ to the edge assigned to c.



Goeritz Matrix Revisited

Definition

Let $D \in \mathbb{R}^2$ be a checkerboard colorable link diagram, and let Γ be a Tait Graph of D. Label the regions of $\mathbb{R}^2 \setminus \Gamma$ by $X_0, X_1, \ldots, X_n$, and let $C_i = \partial X_i \subset \Gamma$. An unreduced Goeritz matrix $\tilde{G}$ of D is given by

$$\tilde{g}_{ij} = \begin{cases} \sum_{e \in C_i \cap C_j} \sigma(e), & i \neq j \\ -\sum_{e \in C_i} \sigma(e), & i = j \end{cases}$$

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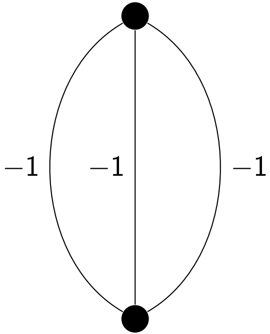
- This definition is clearly equivalent to the previous.

Example

Tait Graph of Trefoil Knot



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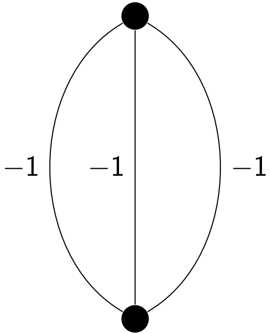

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Matroids

Definition

Let E be a finite set, and $\mathcal{C} \subset \mathcal{P}(E)$ be a set such that:

1. $\emptyset \notin \mathcal{C}$.
2. If $C \in \mathcal{C}$ and $B \subsetneq C$, then $B \notin \mathcal{C}$.
3. If $C, C' \in \mathcal{C}$ with $C \neq C'$ and $e \in C \cap C'$, then there exists $D \subset (C \cup C') \setminus \{e\}$ such that $D \in \mathcal{C}$.

E is a **ground set**, C is a collection of **circuits**, and the pair $M = (E, \mathcal{C})$ is a **matroid**.

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Definitions

$e \in E$ is a **loop** if $\{e\} \in \mathcal{C}$, and e is a coloop if $e \neq C$ for all $C \in \mathcal{C}$. A maximal subset B of E such that $C \not\subset B$ for all $C \in \mathcal{C}$ is a **basis** of M. The **dual matroid** M^* of a matroid M is the matroid such that a set is a basis of M^* if and only if it is the complement of a a basis of M.

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The **bond matroid** $B(\Gamma)$ is the matroid defined by the condition that circuits of $B(\Gamma)$ are minimal cut-sets of Γ. A matroid that is isomorphic to the bond matroid of some graph is said to be a **cographic matroid.**

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- $M(\Gamma)$ and $B(\Gamma)$ are dual.

Example

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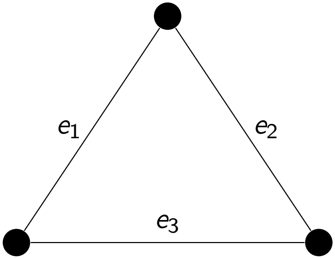

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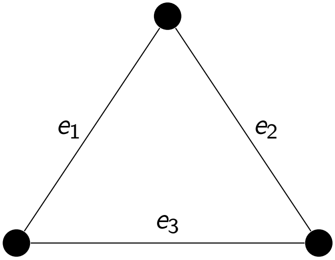

Let $\Gamma = (E, V, \sigma)$ be the Tait graph of the trefoil knot. The cut-sets of Γ are $\mathcal{C} = \{\{e_1, e_2\}, \{e_2, e_3\}, \{e_1, e_3\}\}$. So, $B(\Gamma) = (E, \mathcal{C})$

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Definition

Let $[E]$ denote the $\mathbb{Z}/2\mathbb{Z}$-vector space generated by E, and for $A \subset E$ let $\overline{A} \in [E]$ be $\overline{A} = \sum_{a \in A} a$. Define the **circuit space** of M to be the subspace of $[E]$ generated by $\{\overline{C} | C \in \mathcal{C}\}$. A **2-basis** is a set $\mathcal{B} = \{C_1, \dots, C_n\} \subset \mathcal{C}$ such that $\{\overline{C_1}, \dots, \overline{C_n}\}$ is a basis for the circuit space of M and such that for $C_i, C_j, C_k \in \mathcal{B}$ with $C_i \neq C_j \neq C_k$, $C_i \cap C_j \cap C_k = \emptyset$.

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- All cographic matroids admit a 2-basis.

Example

- Let $M = B(\Gamma) = (E, \mathcal{C})$ where $\mathcal{C} = \{\{e_1, e_2\}, \{e_2, e_3\}, \{e_1, e_3\}\})$, as in the previous example. It is easy to check that $\mathcal{C}$ is a 2-basis of M.

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- More generally, for any bond matroid $B(\Gamma)$ of a graph Γ with vertex set $V = \{v_1, \ldots, v_n\}$, the set $\mathcal{A} = \{A_i, \ldots, A_n\}$ where

$$A_i = \{e \in E : \{e\} \notin M^*, v_i \text{ is an endpoint of } e\},$$

is a 2-basis of $B(\Gamma)$.

Goeritz Matrix Revisited (Again)

Definition

Let $M = (E, \mathcal{C}, \sigma)$ be a cographic matroid and $\mathcal{B} = \{C_1, \ldots, C_n\}$ be a 2-basis of M. A pre-Goeritz matrix $\tilde{G}$ of M is defined by

$$\tilde{g}_{ij} = \begin{cases} \sum_{e \in C_i \cap C_j} \sigma(e), & i \neq j \\ -\sum_{e \in C_i} \sigma(e), & i = j. \end{cases}$$

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- It turns out that cographic matroids are the right setting for us because every symmetric integer matrix is the Goeritz matrix of some signed cographic matroid.

Example

Let $M = B(\Gamma) = (E, \mathcal{C}, \sigma)$ where $\mathcal{C} = \{\{e_1, e_2\}, \{e_2, e_3\}, \{e_1, e_3\}\})$, as in the previous two examples, and σ is induced by Γ. Recall that $\mathcal{C}$ is a 2-basis of M. Then,

$$\tilde{G} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

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μ is Related to The Kauffman Bracket

Theorem 1

Let D be a non-split link diagram, S be a checkerboard surface bounded by D, and G be the Goeritz matrix of D and S. Then,

$$\langle D \rangle = (-A)^{-3w_0(D,S)}\mu[G],$$

where $w_0(D, S)$ is the writhe of the crossings $c \in D$ such that there exists a simple closed curve that intersects S only at c. (These crossings are called S-nugatory.)

Proof Sketch

- Thistlethwaite's polynomial τ is a polynomial of matroids that is defined recursively in terms of contractions and deletions of edges in matroids, similarly to the definition of the Tutte polynomial for graphs.

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- Let G be the Goeritz matrix of a signed cographic matroid M. Using these observations, you can prove by induction on the size of G that $\mu[G]$ is equal to $\tau[M]$ up to a power of $-A$ (which happens to be $w_0(D, S)$ for the bond matroid of a Tait graph).

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- Then if Γ is a Tait graph of a non-split link diagram D, $\tau[B(\Gamma)] = \langle D \rangle$.

Recovering the Jones Polynomial

- Of course if the writhe of the diagram D is known, then you can recover the Jones polynomial

$$J_K(t) = \left[(-A)^{3(w_0(D,S)-w(D))}\mu[G]\right]_{t^{1/2}=A^{-2}}.$$

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$$J_K(t) = \left[(-A)^{3(w_0(D,S)-w(D))}\mu[G]\right]_{t^{1/2}=A^{-2}}.$$

Theorem 2

If, however, the checkerboard surface S of D is orientable, which happens to be equivalent to the condition that the diagonal entries of G are all even, then

$$J_k(t) = \left[(-A)^{3(\sum_{i\leq j} g_{ij})}\mu[G]\right]_{t^{1/2}=A^{-2}}.$$

- This result relies on the homology of S.

Example

We found in our first example that one of the Goeritz matrices of the left-handed trefoil knot is $G = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and that $\mu[G] = A^7 - A^3 - A^{-5}$. Observe that the diagonal entries of G are all even. So, we have that

$$\begin{aligned} J_K(t) &= \left[(-A)^9(A^7 - A^3 - A^{-5})\right]_{t^{1/2}=A^{-2}} \\ &= \left[-A^{16} + A^{12} + A^4\right]_{t^{1/2}=A^{-2}} \\ &= t^{-1} + t^{-3} - t^{-4}. \end{aligned}$$

References

- Boninger, J. (2022). The Jones Polynomial from a Goeritz Matrix. *Bulletin of the London Mathematical Society, 55*(2), 732-755. https://doi.org/10.1112/blms.12753