Some algebraic structures of links: from 0 to ε

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This presentation is an exposition of some common algebraic structures of knots, as well as giving the basic definition of those algebraic jargon in a less formal way to make you less confused. We will eventually return to Jones polynomial.

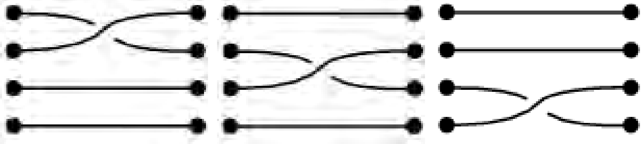
Group

In mathematics we consider the sets and their structures.
Group is a set with one operation "$\cdot$", also we give it extra structures:
1). $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
2). $\exists e \in G$ such that $a \cdot e = e \cdot a = a$
3). $\forall a \in G$, $\exists a^{-1}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$
for all $a, b, c \in G$.

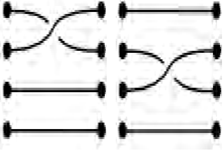
Braid Group

$$B_n = \langle \sigma_1, ..., \sigma_{n-1} | \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_j\sigma_i = \sigma_i\sigma_j, |i-j| > 1 \rangle$$

One can represent diagrammatically the B_n.
Take B_4 for example.



are three generators $\sigma_1, \sigma_2, \sigma_3$ of B_4.
The operation is defined as follow:

$\sigma_1\sigma_2$= 

Alexander's theorem states that every knot or link can be represented as a closed braid, but not one-to-one.

Ring

Ring is the field structure taken out of the commutativity of multiplication and invertibility of multiplication.
A unital ring R is a set, given two operation$(+, \cdot)$ with the following structures:
First layer of structures, as an Abelian group under "+".
1). $a + (b + c) = (a + b) + c$
2). $a + b = b + a$
3). $\exists 0 \in R$ s.t. $a + 0 = a$
4). $\forall a$, $\exists -a \in R$ s.t. $a + (-a) = 0$
Second layer of structures, as a monoid under "$\cdot$".
1). $(a \cdot b) \cdot c = (a \cdot b) \cdot c$
2). $\forall a$, there is $1 \in R$ s.t. $a \cdot 1 = 1 \cdot a = a$
Third layer of structures, distributivity:
1). $a \cdot (b + c) = a \cdot b + a \cdot c$
2). $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$
$\forall a, b, c \in R$

Module and Algebra

Module is the vector space analogue over a ring, it is natural because field is a special ring.
You can have a vector space over a field F.
You can have a module over a ring R. Vector spaces are special module.
Module has structure like vector space. Every element of a module is like a vector.
Module have operation $R \times M \to M$, which sends $(a, u) \to au$.
Satisfying:
1). $a(u + v) = au + av$
2). $(a + b)u = au + bu$
3). $(ab)u = a(bu)$
"Algebra" is when we define the multiplication on the module.
We define the multiplication between any two elements(vectors) in this module(vector spaces), and it should be mapped to another element(vector) in the module(vector space). So, it is a module, and also a ring.

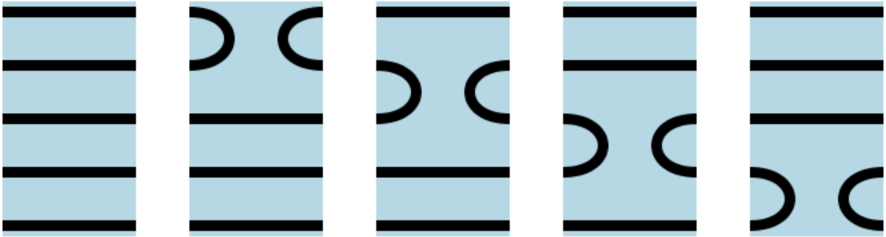
Temperley-Lieb Algebra

This structure is first found in Ising model of statistical mechanics. Jones found and used this to get his Jones polynomial, although from a more abstract approach.
Let $R = \mathbb{Z}[a, a^{-1}]$ be the commutative ring. Fix $\delta = (-a^2 - a^{-2}) \in R$. The Temperley–Lieb algebra $TL_n(\delta)$ is the R-algebra generated by the elements $e_1, e_2, ..., e_{n-1}$, subject to the Jones relations:

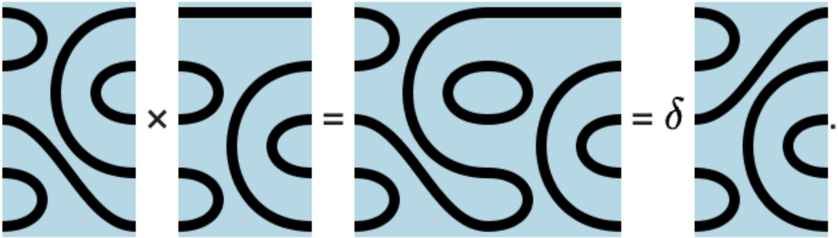
1. $e_i^2 = \delta e_i$ for all $1 \leq i \leq n-1$
2. $e_i e_{i+1} e_i = e_i$ for all $1 \leq i \leq n-2$
3. $e_i e_{i-1} e_i = e_i$ for all $2 \leq i \leq n-1$
4. $e_i e_j = e_j e_i$ for all $1 \leq i, j \leq n-1$ such that $|i-j| \neq 1$

However, one can represent diagrammatically the $TL_n(\delta)$.

For example, the generators of $TL_5(\delta)$ are:

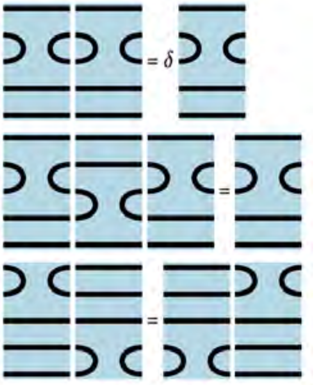


From the left to the right are the unit 1 and the generators e_1, e_2, e_3, e_4
The multiplication on these basis elements can be performed by concatenation like braid group:



Jones's relations

1. $e_i^2 = \delta e_i$ for all $1 \leq i \leq n-1$
2. $e_i e_{i+1} e_i = e_i$ for all $1 \leq i \leq n-2$
3. $e_i e_{i-1} e_i = e_i$ for all $2 \leq i \leq n-1$
4. $e_i e_j = e_j e_i$ for all $1 \leq i, j \leq n-1$ such that $|i-j| \neq 1$



What did Kauffman bracket do?
It actually give a mapping from the braid group to the Temperley-Lieb Algebra, by

$$\begin{array}{rccc} \rho : & B_n & \to & TL_n \\ & \sigma_i & \to & Ae_i + A^{-1}1 \\ & \sigma_i^{-1} & \to & A^{-1}e_i + A1 \end{array}$$

This is actually a homomorphism(representation).

We give a local description of the Jones's polynomial with fixed boundary points.

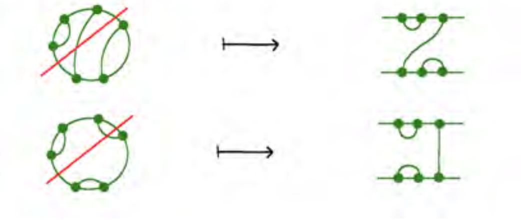
L

By rendering Kauffman's bracket, we eventually get some combinations like:

There are $\frac{1}{n+1}\binom{2n}{n}$ possible combinations, if there are $2n$ boundary points.

Thus, the Kauffman bracket $\langle L\rangle$ takes value in the free $\mathbb{Z}[a, a^{-1}]$-module of rank $N = \frac{1}{n+1}\binom{2n}{n}$.

An equivalent way to represent the same data is by breaking the circle.

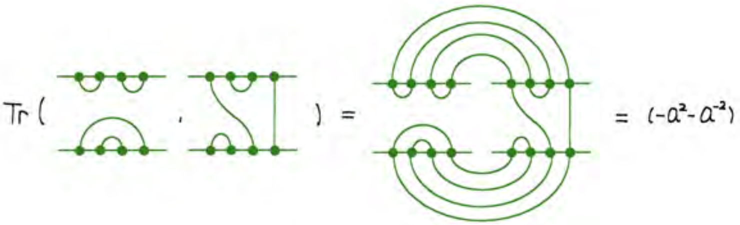


Defining the multiplication by concatenating pictures, we get the Temperley-Lieb Algebra.

To get the Jones Polynomial, we need another operation.

$$Tr : TL_n \times TL_n \to \mathbb{Z}[A, A^{-1}]$$

This is by counting the number of loops in the closure of the Temperley-Lieb algebra, and actually, a bilinear mapping.



How did we get Jones Polynomial out of TL_n?

$$\begin{tikzcd} \rho : B_n \arrow[r] \arrow[dr] & TL_n \arrow[d] \\ & \mathbb{Z}[A, A^{-1}] \end{tikzcd}$$

$Tr \circ \rho(b) = \langle \bar{b} \rangle$, where $\bar{b}$ is the braid closure of b.
So what did we do?
1). Use Alexander's theorem, we represent every link as a braid.
2). We give a representation of braid group to Temperley-Lieb algebra.
3). We get Kauffman bracket by evaluating the trace.
If we define

$$\begin{array}{cccc} \rho' : & B_n & \to & TL_n \\ & \sigma_i & \to & A^{-3}(Ae_i + A^{-1}1) \end{array}$$

We get Jones polynomial directly.

Finitely generated modules over $\mathbb{Z}$

Let M be a free $\mathbb{Z}$-module of rank n, and N be a nonzero submodule of M. Then N is also free, of rank $k \leq n$. Moreover, there is a basis $\{u_1, ..., u_n\}$ of M and scalars $a_1, ..., a_k \in \mathbb{Z}$ with $a_1 \mid a_2 \mid ... \mid a_k$ such that $a_1u_1, ..., a_ku_k$ is a basis in N

This theorem says however the basis $\{u_1, u_2, ..., u_n\}$ of M changes, the scalars $a_1, a_2, ..., a_k$ are uniquely defined up to multiplication by units, in other words, an invariant.

So this theorem yields the smith normal form, saying that for any matrix $A \in \mathbb{Z}_{m\times n}$, there exist invertible matrices $P \in \mathbb{Z}_{m\times m}, Q \in \mathbb{Z}_{n\times n}$ such that $B = PAQ$ has form

$$\begin{pmatrix} a_1 & 0 & \cdots & & 0 \\ 0 & a_2 & & & \\ \vdots & & \ddots & & \vdots \\ & & & a_k & \\ 0 & & \cdots & & 0 \end{pmatrix}$$

Where $a_1 \mid a_2 \mid ... \mid a_k$, are uniquely defined. In vector space, $a_i = 1$

Some words on the Torsion invariant

By rendering same procedures to Goeritz matrix, one get the torsion invariant of $G(D)$.
Naturally, Goeritz matrix defines a module homomorphism from free $\mathbb{Z}$-module M to a submodule.
But why this is also a knot invariant?

Reference

1. Abramsky's book
2. Braid group, Wikipedia
3. Kauffman's report
4. Khovanov's lecture note
5. Tangle, Wikipedia (In Japanese)
6. Temperley-Lieb algebra, Wikipedia
7. Yan's presentation

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