Ribbon Graphs and the Bollobás-Riordan polynomial

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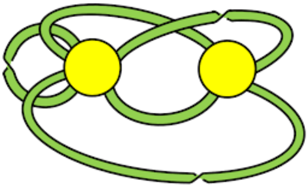
Ribbon Graphs

Definition 1

A ribbion graph G is a surface with a boundary represented by closed topological discs called verticies $V(G)$ and edges $E(G)$, satisfying the following conditions

- the vertices and edges intersect by disjoint line segments
- each such line segment lies on the boundary of only one vertex and only one edge
- every edge contains 2 line segments

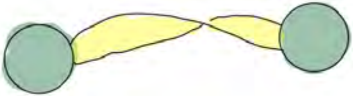


Orientation of Ribbons

Definition 2

An edge of a ribbon graph is orientable if the direction of rotation around a vertex is the same before and after traveling through an edge

Orientable



Non-Orientable

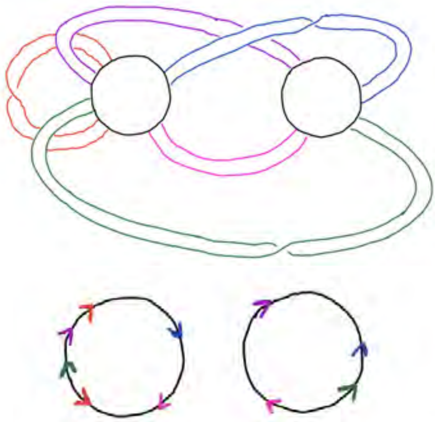
Ribbon Graph Arrow Representation

Definition 3

An arrow representation is the replacement of the edges of a ribbon graph with arrows satisfying the following conditions

- Every edge becomes 2 arrows that are on the 2 vertices the edge connects
- Each arrow is assigned a clockwise or counterclockwise orientation around the vertex it lies on
- If an edge is orientable both arrows associated to the edge have the same orientation. If an edge is non-orientable then both arrows associated with the edge have opposite orientations

Example of Arrow Representation



Ribbon Graph Partial Duality

Definition 4

Let $E' \subset E(G)$. The partial dual $G^{E'}$ of the graph G with respect to edges in E' is given by letting the line segments in E' now belong to $V(G^{E'})$ and adding an arrow to each side of the edges we are taking the dual of. Also, each pair of new arrows has the same orientation.

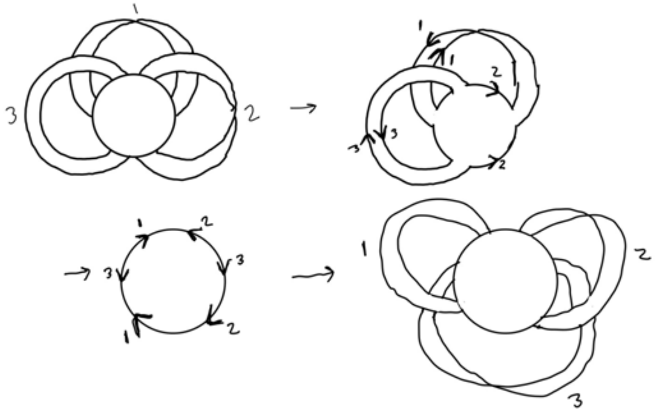
Remark 1

The following are properties of partial duality:

- *If* $e \notin E'$ *then* $(G^{E'})^{\{e\}} = G^{E' \cup e}$
- $(G^{E'})^{E'} = G$
- *Partial duality preserves orientability of ribbon graphs*
- *Partial duality conserves the number of connected components of a ribbon graph*

Example 1

We will look at an example by calculating $G^{\{1,3\}}$ of the following ribbon graph



Partial dual of an edge

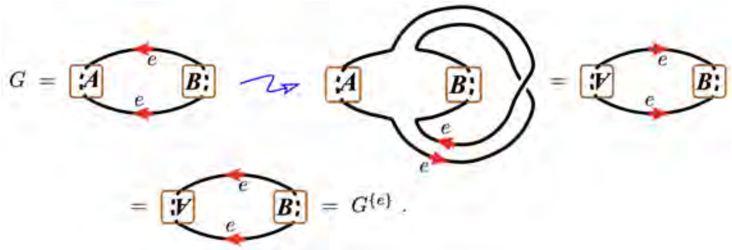
Next we will look at what happens to $G^{\{e\}}$ where e is a single edge.

Case 1: if e is an edge connecting 2 vertices

Case 2: if e is an orientable loop

Partial dual of an edge

Case 3: if e is a non-orientable loop



Contraction-Deletion Operations

Definition 5

The deletion of an edge $G - e$ is defined to be the ribbon graph without the edge e

Definition 6

The contraction of an edge $G/e = G^{\{e\}} - e$ is the deletion of edge e on the partial dual graph of G with respect to edge e.

Contraction on a Single Edge

Case 1: If e connects 2 vertices

Case 1: If e is an orientable loop

Case 1: If e is a non-orientable loop

Signed Ribbon Graph

For a Ribbon graph we can assign a function $\epsilon : E(G) \to \{+1, -1\}$ which assigns a sign to each edge of the Ribbon graph

Definition 7

With a signed ribbon graph we define the following functions that depend on the sign of an edge.

$$x_\epsilon = \begin{cases} x_+ & = 1 \\ x_- & = (\frac{X}{Y})^{1/2} \end{cases} \tag{1}$$

$$y_\epsilon = \begin{cases} y_+ & = 1 \\ y_- & = (\frac{Y}{X})^{1/2} \end{cases} \tag{2}$$

Bollobás-Riordian Polynomial

Definition 8

- $k(G) := \#(\text{connected components of G})$
- $bc(G) := \#(\text{boundary components of G})$
- The rank of a ribbon graph is $r(G) := v(G) - k(G)$
- The nullity of a ribbon graph is $n(G) := e(G) - r(G)$

Definition 9

The Bollobás Riordian polynomial is defined as follows where F is a sub-graph of G with the same number of vertices.

$$R_G(X, Y, Z) = \sum_{F \subseteq G} \left(\prod_{\epsilon \in F} x_\epsilon \right) \left(\prod_{\epsilon \notin F} y_\epsilon \right) X^{r(G) - r(F)} Y^{n(F)} Z^{k(F) - bc(F) + n(F)}$$

Partial Duality and Bollobás-Rioridan Polynomial

Remark 2

For signed ribbon graphs we alternate the signs of the dual edges in the partial dual graph. So for $e \in E'$ we have $\epsilon_G(e) = -\epsilon_{G\{E'\}}(e)$

Theorem 10

The restriction of the polynomial

$$X^{k(G)} Y^{v(G)} Z^{v(G)+1} R_G(X, Y, Z)$$

to the surface $XYZ^2 = 1$ is invariant under partial duality. In other words, for any choice of edges E', if $G' = G^{\{E'\}}$ then

$$X^{k(G)} Y^{v(G)} Z^{v(G)+1} R_G(X, Y, Z)|_{XYZ^2=1}$$

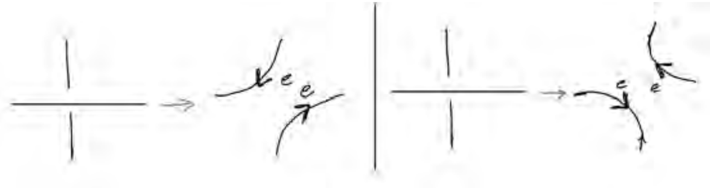
$$= X^{k(G')} Y^{v(G')} Z^{v(G')+1} R_{G'}(X, Y, Z)|_{XYZ^2=1}$$

Ribbon Graphs and Knots

Definition 11

For a Kauffman state s of a knot K (possibly virtual) we define the ribbon graph G_s as follows.

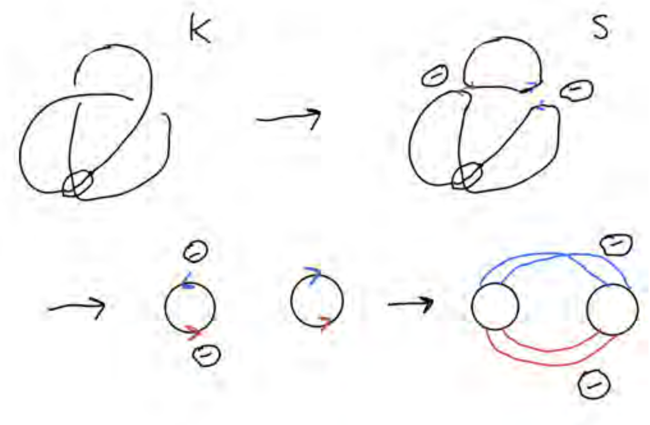
1. At every A splitting add a positive signed ribbon with arrow representation as depicted below.
2. At every B splitting add a negative signed ribbon with arrow representation as depicted below.



A-split B-split

Example

We will Look at an example of the Virtual Trefoil with the state where we take all crossings to be B-splittings so all edges are negative



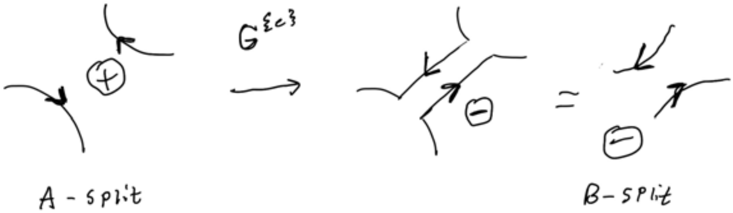
Partial Dual Graphs of Ribbon Graph of a Knot

Theorem 12

Let $G_s^{\{E'\}}$ be the dual graph with respect to the set of edges E' of G_s. Let C' be the set of classical crossings of the knot that become the edges E' of G_s. if G_s is obtained from a Kauffman state s then $G_s^{\{E'\}} = G_{s'}$ where s' is the Kauffman state obtained by switching all A-splittings of crossings in C' to B-Splittings and vice versa.

Proof of Theorm

Proof: Because of the equation, If $e \notin E'$ then $(G^{E'})^{\{e\}} = G^{E' \cup e}$, we need to only prove this theorem for the case where $E' = \{e\}$. Also, Since we have that $(G^{\{e\}})^{\{e\}} = G$ if $G^{\{e\}}$ will cause an A-splitting to switch to a B-splitting then we will also have that it will cause B-splittings to switch to A-splittings. Depicted below shows this partial dual when applied to an A-splitting.



Retrieving the Kauffman Bracket

In the paper [CV], for a specific state s' they called the Seifert state it was shown that:

$$\langle L\rangle(A,B,d) = A^{e(G_{s'})}\left(X^{k(G_{s'})}Y^{v(G_{s'})}Z^{v(G_{s'})+1}R_{G_{s'}}(X,Y,Z)\Big|_{X=\frac{Ad}{B},Y=\frac{Bd}{A},Z=\frac{1}{d}}\right)$$

So from the previous 2 theorems we have for any state s of a Knot

$$\langle L\rangle(A,B,d) = A^{e(G_{s})}\left(X^{k(G_{s})}Y^{v(G_{s})}Z^{v(G_{s})+1}R_{G_{s}}(X,Y,Z)\Big|_{X=\frac{Ad}{B},Y=\frac{Bd}{A},Z=\frac{1}{d}}\right)$$

Recursive Relationships for the Bollobás-Riordian Polynomial

For any Bollobás-Riordian polynomial the following recursive relationships hold.

- If e is an ordinary edge. (not a loop or bridge)

$$R_G = x_\epsilon R_{G/e} + y_\epsilon R_{G-e}$$

- If e is a bridge

$$R_G = (x_\epsilon + X y_\epsilon) R_{G/e}$$

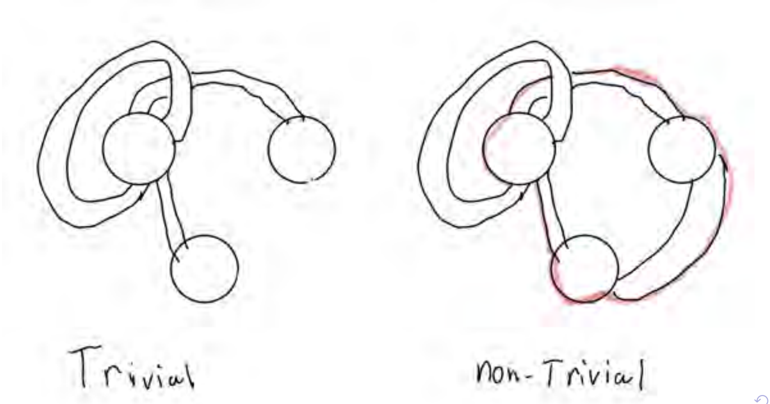
- If e is a non-orientable loop

$$YZ x_\epsilon R_G + y_\epsilon R_{G-e}$$

Trivial loops

Definition 13

A loop of a Ribbon Graph is trivial if there does not exist a path from one side of the loop to the other, such that the path is disjoint from the loop.



Recursive Relationships for Orientable Loops

The following are recursive relationships for loops of the Bollobás-Riordian polynomial

- If e is a trivial orientable loop

$$(R_G = Yx_\epsilon + y_\epsilon)R_{G-e}$$

- If e is a non-trivial orientable loop

$$R_G\Big|_{XYZ^2=1} = y_\epsilon R_{G-e} + Y^2Z^2x_\epsilon R_{G/e}\Big|_{XYZ^2=1}$$

Kauffman bracket polynomial

Definition 14

let $\langle L_G \rangle(A)$ non-normalized Jones polynomial given by the Kauffman bracket of a ribbon graph and letting $B = A^{-1}$ and $d = -A^2 - A^{-2}$

Definition 15

$$A_\epsilon = \begin{cases} A_+ &= A \\ A_- &= A^{-1} \end{cases}$$

Kauffman bracket recursive relations

We obtain the following recursive relationships by substitution of the previous recursive relationships into the Kauffman bracket polynomial.

- If e is a non-orientable loop, nontrivial orientable loop, or and ordinary edge.

$$\langle L_G \rangle = A_\epsilon \langle L_{G-e} \rangle + A_{-\epsilon} \langle L_{G/e} \rangle$$

- If e is a bridge

$$\langle L_G \rangle = (-A_\epsilon)^3 \langle L_{G/e} \rangle$$

- if e is a trivial orientable loop

$$\langle L_G \rangle = (-A_{-\epsilon})^3 \langle L_{G-e} \rangle$$

Open Question

- The recursive relationships for the Kauffman bracket polynomial when restricted to planar ribbon graphs guarantee that we will never have a nontrivial or non-orientable loop.
- Also, all classical knots have a planar ribbon graph corresponding to one of their Kauffman brackets.
- The remaining recursive relationships are the exact same relationships that define Thistlethwaite's polynomial as defined in [BO]
- Is there a way to extend the Idea of [BO] to some a matrix that can recursively give the Kauffman bracket and Jones polynomial of an arbitrary ribbon graph and hence an arbitrary virtual knot?

References

CV S. Chmutov, J. Voltz, *Thistlethwaite's theorem for virtual links*, Journal of Knot Theory and Its Ramifications, 17(10) (2008) 1189-1198; preprint arXiv:math.GT/0704.1310

CH S. Chmutov,*Generalized duality for graphs on surfaces and the signed Bollobás-Riordian polunimial*, Journal of Combinatorial Theory, Ser. B 99(3) (2009) 617-638; preprint arXiv:math.C0/0711.3490.

BO Boninger, J. (2022). The Jones Polynomial from a Goeritz Matrix. Bulletin of the London Mathematical Society, 55(2), 732-755. https://doi.org/10.1112/blms.12753