Goeritz Matrix and Knot Coloring

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Checkerboard Coloring of Link Diagram

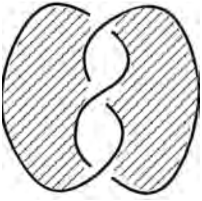
Definition 1

Let D be a link diagram in $\mathbb{R}^2$. A *checkerboard coloring* of D is a coloring of $\mathbb{R}^2 \setminus D$ by $\{0, 1\}$ such that no arc has two monochromatic sides.

Theorem 2

Every classical link is checkerboard colorable.

Proof: Jordan curve theorem.



Checkerboard coloring of Virtual Links

Note that not all Virtual links can be checkerboard colored

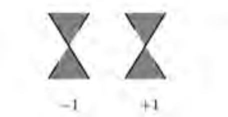
Checkerboard Colorable

Not Checkerboard Colorable

Goeritz Matrix

Definition 3

The sign of a crossing $\eta(c)$:



Definition 4

Take a checkerboard colored link diagram D with the unbounded region shaded. Enumerate the shaded regions by $1, 2, \ldots, n$. The *pre-Goeritz matrix* of D is a $n \times n$ matrix (g_{ij}):

$$g_{ij} = \begin{cases} \sum_{\text{crossings between } i,j} \eta(c), & i \neq j \\ -\sum_{k \neq i} g_{ik}, & i = j \end{cases}$$

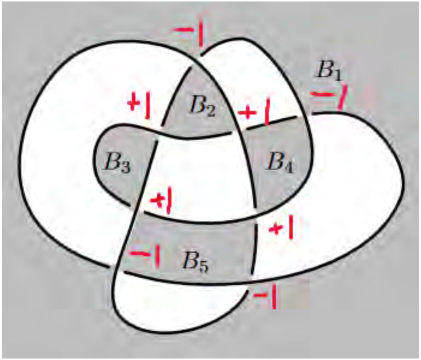
Goeritz Matrix

Definition 5

Since the sum of rows of pre-Goeritz matrix is always 0 by definition, we can always delete one row and one column without losing any information. What we obtain is called *Goeritz matrix G*. Usually we delete the row and column corresponding to the unbounded region.

Example

Take the knot 8_{19} and assign each crossing ± 1:



Example

Its pre-Goeritz matrix is

$$\vec{G} = \begin{bmatrix} 4 & -1 & 0 & -1 & -2 \\ -1 & -1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 1 \\ -1 & 1 & 0 & -1 & 1 \\ -2 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Delete the first row and column, we get the Goeritz matrix:

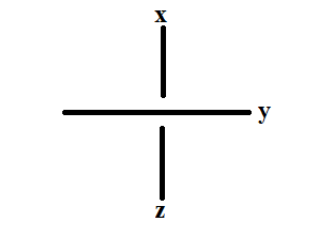
$$G = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Fox Coloring and Dehn Coloring

An arc in the link diagram starts and ends where it goes underneath.

Definition 6

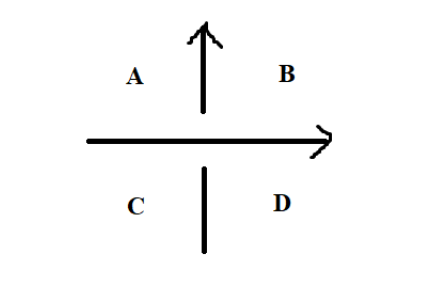
A *Fox coloring* is a coloring of a link diagram by $\mathbb{Z}/p\mathbb{Z}$ (or $\mathbb{Z}/n\mathbb{Z}$) such that at each crossing the equation $x + z = 2y$ holds:



Fox Coloring and Dehn Coloring

Definition 7

Given an oriented link diagram D, a *Dehn coloring* is a coloring of $\mathbb{R}^2 \setminus D$ by $\mathbb{Z}/p\mathbb{Z}$. At each crossing it follows the equation $A + B = C + D$:



Fox Coloring and Dehn Coloring

Remark 1

The following properties hold for both Fox coloring and Dehn coloring:

- *Any $k \in \mathbb{Z}/n\mathbb{Z}$ gives a monochromatic coloring.*
- *Adding up two colorings gives a coloring.*
- *Multiplying a coloring by k gives a coloring.*

In conclusion:

Theorem 8

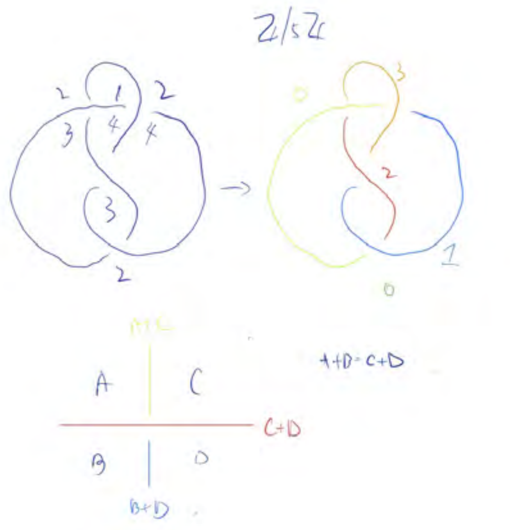
Every Fox/Dehn coloring of a link diagram is a $\mathbb{Z}/n\mathbb{Z}$ module (a vector space when n is prime).

Fox Coloring and Dehn Coloring

We can quotient out the monochromatic colorings, which can be done by setting one arc/the unbounded region to be 0. There is a one-to-one relation between the quotients of two colorings:

Each Dehn coloring gives a Fox coloring: assign the arc with the sum of the colors on both sides. The coloring is well defined and satisfies the rule $x + z = 2y$.

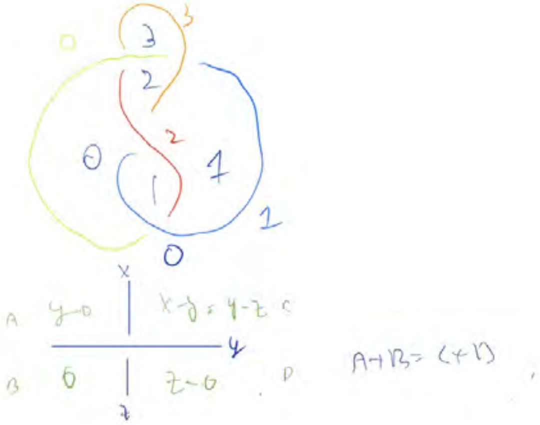
Fox Coloring and Dehn Coloring



Fox Coloring and Dehn Coloring

Each Fox coloring gives a Dehn coloring: assign 0 to the unbounded region. For the rest region, draw a line connecting the center of the region with the interior of the unbounded region. Once the line enters from region R_1 to region R_2 crossing arc c, color R_2 by color of c minus the color of R_1. We can prove the coloring is well defined and satisfying the rule $A + B = C + D$.

Fox Coloring and Dehn Coloring



Fox Coloring and Dehn Coloring

Warning

The two maps are not inverses of each other.

In fact, the Fox coloring and Dehn coloring can be given via Goeritz matrix:

Theorem 9

The quotiented Dehn coloring, the quotiented Fox coloring, and the solution space of Goeritz matrix are isomorphic as $\mathbb{Z}/n\mathbb{Z}$ *modules.*

Alexander Numbering

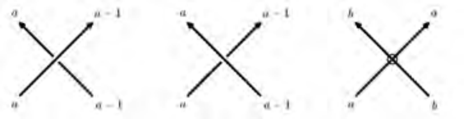
Remark 2

Checkerboard coloring is a $\mathbb{Z}/2\mathbb{Z}$ *Dehn coloring.*

Checkerboard coloring can be generalized to virtual links via Alexander numbering(mod 2):

Definition 10

For an oriented virtual link diagram, an *Alexander numbering* is a coloring of its semi-arcs(arc between two classical crossings) given by the following relation:



Alexander Numbering

Theorem 11

The following are equivalent:

- *A virtual link L has a diagram D which is Alexander numberable(mod 2).*
- *(Σ, L) has a (oriented) spanning surface, where (Σ, L) is the embedding of L in the Carter surface corresponding to D.*
- *$\Sigma \backslash D'$ is checkerboard colorable, where D' is the link diagram on the Carter surface Σ corresponding to D.*
- *D has no "windmill" crossing.*

Pseudo Goeritz Matrix

N. Kamada has generalized Goeritz matrix to virtual link diagrams:

Definition 12

The first and second local index η:

fist local index second local index

Pseudo Goeritz Matrix

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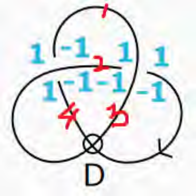
Definition 13

For an oriented virtual diagram D with n semi-arcs(arcs between two classical crossings), the *pseudo Goeritz matrix* $G = (g_{ij})$ is an $n \times n$ matrix:

$$g_{ij} = \begin{cases} \sum_{\text{crossings between i,j}} \eta, & i \neq j \\ -\sum_{k \neq i} g_{ik}, & i = j \end{cases}$$

Example

The knot $2-1$ with second indices:



Then its pseudo Goeritz matrix is

$$\begin{bmatrix} -2 & 0 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

Fox coloring matrix

Definition 14

With the coloring equations we defined earlier we can form a matrix where each row represents a crossing and each column a arc of the link diagram. We will call this the pre-Coloring matrix $\overrightarrow{C}$.

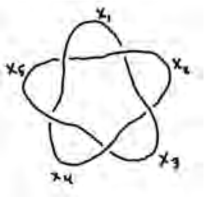
Definition 15

The Coloring matrix C of a Link diagram is obtained by deleting any row and any column of the pre-Coloring matrix $\overrightarrow{C}$

Remark 3

Any non-constant vector x in $\mathbb{Z}/n\mathbb{Z}$ such that $\overrightarrow{C}x = 0$ corresponds with a non-trivial fox n-coloring of the Link diagram

Example



The coloring equations are:

$$
\begin{aligned}
x_1 + x_3 &= 2x_2 \\
x_2 + x_4 &= 2x_3 \\
x_3 + x_5 &= 2x_4 \\
x_4 + x_1 &= 2x_5 \\
x_5 + x_2 &= 2x_1
\end{aligned}
$$

Example

These equations form the pre-Coloring matrix

$$\vec{C} = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \\ -2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

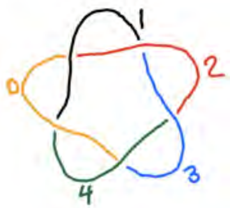
By deleting the first row and column we get the coloring matrix

$$C = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Example

To find an example of a coloring we can see the following equality holds in $\mathbb{Z}/5\mathbb{Z}$ and leads the the 5-coloring depicted below.

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \\ -2 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



Link determinants

Definition 16

The Link determinant of a Link L is the determinant of the Coloring Matrix C

Theorem 17

The Link determinant of a Link is well defined and $|\det(C)|$ *is a Knot Invariant*

Theorem 18

for a prime number p a link has a nontrivial p-coloring iff p divides the link determinate

Example

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Lets look at the pre-Coloring matrix for the knot 8_{19}

$$\vec{C} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \end{bmatrix}$$

Example

By deleting the top row and left column we get the Coloring matrix:

$$C = \begin{bmatrix} 1 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 \end{bmatrix}$$

Taking the determinant we get $\det(C) = 3$. So we can conclude the only non-trivial prime coloring of this knot is a 3-coloring.

Knot determinant's and the Goeritz matrix

Theorem 19

The absolute value of the determinant of the Goeritz Matrix of a link L is equal to the absolute value of Link determinant for a link L. $|\det(G)| = |\det(C)|$

With this theorem we can calculate the knot determinate faster because the Goeritz matrix is always smaller than the coloring matrix of a knot.

Example

We calculated the Goeritz matrix for 8_{19} on an earlier slide which was:

$$G = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

So when we take the determinant of this matrix we get $\det(G) = -3$. We can compare this to the determinant of the coloring matrix we calculated earlier and we can see that we have $|\det(G)| = 3 = |\det(C)|$.