# Revisiting the Conway Polynomial for Virtual Links

Pranav Jois

The Ohio State University

10 July 2023

# A Review of the Conway Polynomial

# A Review of the Conway Polynomial

Let $D$ be a virtual link diagram with $n \geq 1$ classical crossings $c_1, ..., c_n$. Define

$$M_+ := \begin{pmatrix} 1-x & -y \\ -xy^{-1} & 0 \end{pmatrix} \quad \text{and} \quad M_- = \begin{pmatrix} 0 & -x^{-1}y \\ -y^{-1} & 1-x^{-1} \end{pmatrix}$$

# A Review of the Conway Polynomial

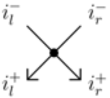
Let $D$ be a virtual link diagram with $n \geq 1$ classical crossings $c_1, ..., c_n$. Define

$$M_+ := \begin{pmatrix} 1-x & -y \\ -xy^{-1} & 0 \end{pmatrix} \quad \text{and} \quad M_- = \begin{pmatrix} 0 & -x^{-1}y \\ -y^{-1} & 1-x^{-1} \end{pmatrix}$$

For $i = 1, ..., n$, let $M_i := M_+$ if $c_i$ is positive, and let $M_i = M_-$ otherwise. Define the $2n \times 2n$ matrix as a block matrix by $M = \text{diag}(M_1, ..., M_n)$.

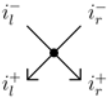
# A Review of the Conway Polynomial

Consider the graph belonging to the virtual link diagram where the virtual crossings are ignored. Subdivide the edges into two half-edges and label them at each vertex $v_i$ as follows:



# A Review of the Conway Polynomial

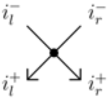
Consider the graph belonging to the virtual link diagram where the virtual crossings are ignored. Subdivide the edges into two half-edges and label them at each vertex $v_i$ as follows:




A permutation of $\{1, ..., n\} \times \{l, r\}$ is given by the assignment $(i, a) \longmapsto (j, b)$ if the half-edges $i_a^+$ and $j_b^-$ belong to the same edge of the virtual diagram's graph. Let $P$ denote the corresponding $2n \times 2n$ permutation matrix.

# A Review of the Conway Polynomial

Consider the graph belonging to the virtual link diagram where the virtual crossings are ignored. Subdivide the edges into two half-edges and label them at each vertex $v_i$ as follows:




A permutation of $\{1, ..., n\} \times \{l, r\}$ is given by the assignment $(i, a) \longmapsto (j, b)$ if the half-edges $i_a^+$ and $j_b^-$ belong to the same edge of the virtual diagram's graph. Let $P$ denote the corresponding $2n \times 2n$ permutation matrix.

We can define $Z_D(x, y) = (-1)^{w(D)}\det(M - P)$ and $\tilde{Z}_D(x, y) = x^{-N} Z_D(x, y)$ where $N$ is the lowest exponent in the variable $x$ of $Z_D(x, y)$. $\tilde{Z}_D$ is an invariant of virtual links.

# Invariants of Four-Regular Graphs

# Invariants of Four-Regular Graphs

Definition: An Eulerian orientation of a graph is an orientation such that for each vertex $v$, the number of edges with initial end $v$ equals the number of edges with terminal end $v$.

# Invariants of Four-Regular Graphs

Definition: An Eulerian orientation of a graph is an orientation such that for each vertex $v$, the number of edges with initial end $v$ equals the number of edges with terminal end $v$.

Let $G = (V, E)$ be a 4-regular graph provided with an Eulerian orientation. A *labelling* of $G$ is any mapping $f : G \to \{1, 2\}$ such that $f^{-1}(1)$ and $f^{-1}(2)$, define subgraphs which are Eulerian provided the orientation inherited from $G$.

# Invariants of Four-Regular Graphs

Definition: An Eulerian orientation of a graph is an orientation such that for each vertex $v$, the number of edges with initial end $v$ equals the number of edges with terminal end $v$.

Let $G = (V, E)$ be a 4-regular graph provided with an Eulerian orientation. A *labelling* of $G$ is any mapping $f : G \to \{1, 2\}$ such that $f^{-1}(1)$ and $f^{-1}(2)$, define subgraphs which are Eulerian provided the orientation inherited from $G$.

Define $\mathscr{L}(G)$ to be the set of all labellings of $G$.

# Invariants of Four-Regular Graphs (cont.)

Divide each edge $e$ into two half-edges $e^+$ and $e^-$, where $e^+$ is the incident end of $e$ and $e^-$ is the terminal end of $e$.

# Invariants of Four-Regular Graphs (cont.)

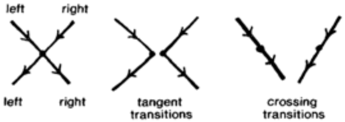
Divide each edge $e$ into two half-edges $e^+$ and $e^-$, where $e^+$ is the incident end of $e$ and $e^-$ is the terminal end of $e$.

Given a vertex $v$, a *transition* at $v$ is an ordered pair $(e_1^+, e_2^-)$, where $e_1$ has terminal end $v$ and $e_2$ has initial end $v$. Each half-edge will be arbitrarily assigned a type (left or right) s.t. at every vertex $v$, the two initial half-edges have opposite types and likewise for terminal edges. From this, one can define *tangent* and *initial* transitions (denoted by $(\text{type}(e_1^+), \text{type}(e_2^-))$.

# Invariants of Four-Regular Graphs (cont.)

Divide each edge $e$ into two half-edges $e^+$ and $e^-$, where $e^+$ is the incident end of $e$ and $e^-$ is the terminal end of $e$.

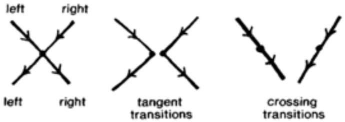
Given a vertex $v$, a *transition* at $v$ is an ordered pair $(e_1^+, e_2^-)$, where $e_1$ has terminal end $v$ and $e_2$ has initial end $v$. Each half-edge will be arbitrarily assigned a type (left or right) s.t. at every vertex $v$, the two initial half-edges have opposite types and likewise for terminal edges. From this, one can define *tangent* and *initial* transitions (denoted by $(\text{type}(e_1^+), \text{type}(e_2^-))$.



# Invariants of Four-Regular Graphs (cont.)

Divide each edge $e$ into two half-edges $e^+$ and $e^-$, where $e^+$ is the incident end of $e$ and $e^-$ is the terminal end of $e$.

Given a vertex $v$, a *transition* at $v$ is an ordered pair $(e_1^+, e_2^-)$, where $e_1$ has terminal end $v$ and $e_2$ has initial end $v$. Each half-edge will be arbitrarily assigned a type (left or right) s.t. at every vertex $v$, the two initial half-edges have opposite types and likewise for terminal edges. From this, one can define *tangent* and *initial* transitions (denoted by $(\text{type}(e_1^+), \text{type}(e_2^-))$.




For a vertex $v$ and $p, q \in \{l, r\}$, we can denote by $\theta_{p,q}(v)$ the transition at $v$ of type $(p, q)$.

# Invariants of Four-Regular Graphs (cont.)

For every labelling $f \in \mathscr{L}(G)$, let $H(v, f)$ be the set of half-edges incident to $v$ which belong to an edge $e$ with $f(e) = 1$. If $|H(v, f)| = 2$, then the two elements of $H(v, f)$ form a transition at $v$.

# Invariants of Four-Regular Graphs (cont.)

For every labelling $f \in \mathscr{L}(G)$, let $H(v, f)$ be the set of half-edges incident to $v$ which belong to an edge $e$ with $f(e) = 1$. If $|H(v, f)| = 2$, then the two elements of $H(v, f)$ form a transition at $v$.

Now, we can define the quantity $\langle v|f \rangle$. If $|H(v, f)| = 0$, $\langle v|f \rangle = 1$, if $|H(v, f)| = 2$, $\langle v|f \rangle = \theta_{p,q}(v)$ (where $\theta_{p,q}(v)$ corresponds to the transition defined by $H(v, f)$), and if $|H(v, f)| = 4$, $\langle v|f \rangle = \theta_{l,l}(v)\theta_{r,r}(v)$ $-\theta_{l,r}(v)\theta_{r,l}(v) := \Delta(v)$.

# Invariants of Four-Regular Graphs (cont.)

For every labelling $f \in \mathscr{L}(G)$, let $H(v, f)$ be the set of half-edges incident to $v$ which belong to an edge $e$ with $f(e) = 1$. If $|H(v, f)| = 2$, then the two elements of $H(v, f)$ form a transition at $v$.

Now, we can define the quantity $\langle v|f \rangle$. If $|H(v, f)| = 0$, $\langle v|f \rangle = 1$, if $|H(v, f)| = 2$, $\langle v|f \rangle = \theta_{p,q}(v)$ (where $\theta_{p,q}(v)$ corresponds to the transition defined by $H(v, f)$), and if $|H(v, f)| = 4$, $\langle v|f \rangle = \theta_{l,l}(v)\theta_{r,r}(v)$ $-\theta_{l,r}(v)\theta_{r,l}(v) := \Delta(v)$.

Now, we can split each vertex of $G$ into two vertices of degree 2, each one being incident to the two half-edges of a tangent transition. The resultant circuits are called *Seifert circuits*. Let $s(G)$ be the number of these circuits, and let $s(G, f, i)$ for $i \in 1, 2$ the number of Seifert circuits of the subgraph defined by $f^{-1}(i)$.

# Construction of Determinantal Invariant

# Construction of Determinantal Invariant

Let us consider the partitition function

$$Z(G) = \sum_{f \in \mathscr{L}(G)} (-1)^{s(G,f,2)} \prod_{v \in V} \langle v | f \rangle$$

# Construction of Determinantal Invariant

Let us consider the partitition function

$$Z(G) = \sum_{f \in \mathscr{L}(G)} (-1)^{s(G,f,2)} \prod_{v \in V} \langle v|f \rangle$$

Let $\varepsilon_{p,q} = 2\delta_{p,q} - 1$. If $n = |V|$, consider the $2n \times 2n$ matrix indexed by $\{1, ..., n\} \times \{l, r\}$ (which we can call $A$) given by $A = T - P$, where, for all $i, j \in \{1, ..., n\}$ and $p, q \in \{l, r\}$:

$$T_{(i,p),(j,q)} = \delta_{i,j}\varepsilon_{p,q}\theta_{p,q}(v_i)$$

$$P_{(i,p),(j,q)} = \begin{cases} -1 & \text{if } h^-_{i,p}, h^+_{j,q} \text{are the two half edges of the same edge} \\ 0 & \text{otherwise} \end{cases}$$

# Construction of Determinantal Invariant

Let us consider the partitition function

$$Z(G) = \sum_{f \in \mathscr{L}(G)} (-1)^{s(G,f,2)} \prod_{v \in V} \langle v|f \rangle$$

Let $\varepsilon_{p,q} = 2\delta_{p,q} - 1$. If $n = |V|$, consider the $2n \times 2n$ matrix indexed by $\{1, ..., n\} \times \{l, r\}$ (which we can call $A$) given by $A = T - P$, where, for all $i, j \in \{1, ..., n\}$ and $p, q \in \{l, r\}$:

$$T_{(i,p),(j,q)} = \delta_{i,j}\varepsilon_{p,q}\theta_{p,q}(v_i)$$

$$P_{(i,p),(j,q)} = \begin{cases} -1 & \text{if } h^-_{i,p}, h^+_{j,q} \text{are the two half edges of the same edge} \\ 0 & \text{otherwise} \end{cases}$$

Proposition: $Z(G) = \det(A)$

# Outline of Proof of Invariance

# Outline of Proof of Invariance

We start by considering invariance of models of the form

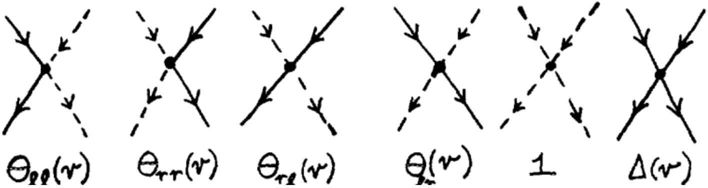
$$Z(G) = \sum_{f \in \mathscr{L}(G)} (-1)^{s(G,f,2)} \prod_{v \in V} \langle v | f \rangle$$

# Outline of Proof of Invariance

We start by considering invariance of models of the form

$$Z(G) = \sum_{f \in \mathscr{L}(G)} (-1)^{s(G,f,2)} \prod_{v \in V} \langle v | f \rangle$$

Recall that for a plane graph $G$, at any vertex $v$, there are six different local labelling configurations

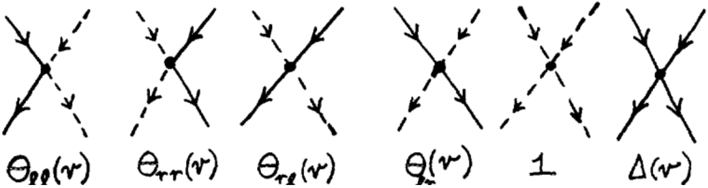


# Outline of Proof of Invariance

We start by considering invariance of models of the form

$$Z(G) = \sum_{f \in \mathscr{L}(G)} (-1)^{s(G,f,2)} \prod_{v \in V} \langle v | f \rangle$$

Recall that for a plane graph $G$, at any vertex $v$, there are six different local labelling configurations




Furthermore, in an oriented classical link diagram, each crossing is either positive or negative. So we consider transitions $\theta_{p,q}(s)$ where $s \in \{+, -\}$.

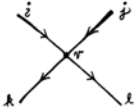
# Outline of Proof of Invariance (cont.)

We can now consider the matrix $R(s)$ which compactly encodes the values of $\langle v|f\rangle$ for every vertex $v$ of type $s$ and labelling $f$.

# Outline of Proof of Invariance (cont.)

We can now consider the matrix $R(s)$ which compactly encodes the values of $\langle v|f\rangle$ for every vertex $v$ of type $s$ and labelling $f$.

Index the matrix $R(s)$ by $\{1,2\} \times \{1,2\}$ and let the entries be equal to the value of $R^{k,l}_{i,j}(s) = \langle v|f_0\rangle$, where $f_0$ is a labelling of $G$ such that at the vertex $v$, the incident and terminating edges take on the values $i, j, k, l \in \{1,2\}$ as follows:

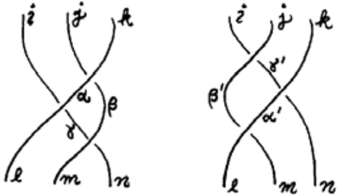



From this, we can deduce that

$$R(s) = \begin{pmatrix} \Delta(s) & 0 & 0 & 0 \\ 0 & \theta_{l,l}(s) & \theta l, r(s) & 0 \\ 0 & \theta_{r,l}(s) & \theta_{r,r}(s) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

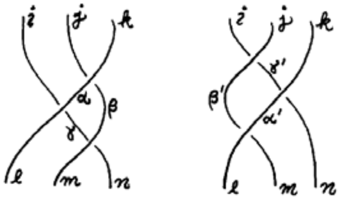
# Outline of Proof of Invariance (cont.)

Consider the third Reidemeister move with positive crossings. Suppose $i, j, k, l, m, n$ are fixed, and $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ can vary.



## Outline of Proof of Invariance (cont.)

Consider the third Reidemeister move with positive crossings. Suppose $i, j, k, l, m, n$ are fixed, and $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ can vary.




One can see that all labellings in $\mathscr{L}(G)$ that preserve the fixed edge-values outside the triangle have the same value of $s(G, f, 2)$ in both configurations. Thus, for invariance of the partition function, we require that

$$\sum_{\alpha,\beta,\gamma} R^{\alpha,\beta}_{j,k}(+) R^{i,\alpha}_{l,\gamma}(+) R^{m,n}_{\gamma,\beta}(+) = \sum_{\alpha,\beta,\gamma} R^{\beta',\gamma'}_{l,j}(+) R^{\alpha',n}_{\gamma',k}(+) R^{l,m}_{\beta',\alpha'}(+)$$

# Outline of Proof of Invariance (cont.)

Note that this equation can be reduced to the matrix equation $STS = TST$ where $S$ and $T$ are cleverly defined $8 \times 8$ matrices. This is related to the Yang-Baxter equation in statistical mechanics.

# Outline of Proof of Invariance (cont.)

Note that this equation can be reduced to the matrix equation $STS = TST$ where $S$ and $T$ are cleverly defined $8 \times 8$ matrices. This is related to the Yang-Baxter equation in statistical mechanics.

From here, one can check that the following assignment of $\theta_{p,q}$ in $\mathbf{Z}[x, y, x^{-1}, y^{-1}]$ satisfies the given condition:

$$\theta_{l,l}(+) = 1 + x, \quad \theta_{r,r}(+) = 0, \quad \theta_{l,r}(+) = y, \quad \theta_{r,l}(+) = -xy^{-1}$$

# Outline of Proof of Invariance (cont.)

Note that this equation can be reduced to the matrix equation $STS = TST$ where $S$ and $T$ are cleverly defined $8 \times 8$ matrices. This is related to the Yang-Baxter equation in statistical mechanics.

From here, one can check that the following assignment of $\theta_{p,q}$ in $\mathbf{Z}[x, y, x^{-1}, y^{-1}]$ satisfies the given condition:

$$\theta_{l,l}(+) = 1 + x, \quad \theta_{r,r}(+) = 0, \quad \theta_{l,r}(+) = y, \quad \theta_{r,l}(+) = -xy^{-1}$$

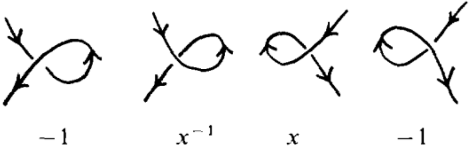
Using a similar analysis of the RII move, one can deduce that

$$\theta_{l,l}(-) = 0, \quad \theta_{r,r}(-) = 1 + x^{-1}, \quad , \theta_{l,r}(-) = -x^{-1}y, \quad \theta_{r,l}(-) = y^{-1}$$

is also required to satisfy the corresponding conditions that follow from performing the RII operation.

# A Note on Reidemeister Move I

The polynomial as presented is *not* invariant under RI (This can be seen using the same technique utilized earlier). Creating a new crossing by RI multiplies the entire polynomial by a constant factor given by the following:



# References

J. Sawollek, On Alexander-Conway Polynomials for Virtual Knots and Links, arXiv Preprint (2001).

F. Jaeger, L.H. Kauffman, and H. Saleur, The Conway Polynomial in $R^3$ and in Thickened Surfaces: A New Determinant Formulation, *J. Combin. Theory Ser. B* 61 (1994), 237–259.