Two Generalizations of Fox n-Colorability via Polynomials and Quandles

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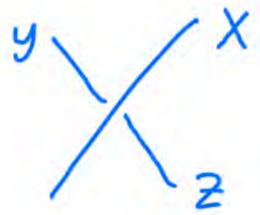
Recall: Fox n-Coloring

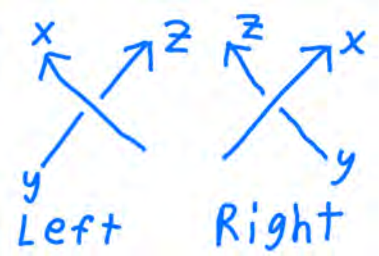
Definition

For an integer $n \geq 3$, a knot diagram is Fox n-colorable if its arcs can be labelled by $\mathbb{Z}/n\mathbb{Z}$ such that at least two colors are used, and at each crossing, the equation $2x - y - z = 0$ holds.



Polynomial Coloring

We will first investigate polynomial generalizations of the crossing equation, following the exposition in the 2022 paper by He, Ho, Kalir, Miller, and Zevenbergen.



We will work with oriented knots to distinguish between the two undercrossings and to have two equations for left- and right-handed crossings.

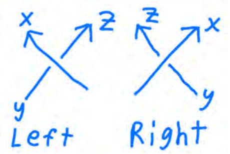
Polynomial Coloring

Definition

Let R be a ring with $|R| \geq 2$ and $f, g \in R[x, y, z]$. Let D be an oriented knot diagram. We say D is $(\boldsymbol{f}, \boldsymbol{g})_{\boldsymbol{R}}$ **colorable** if each arc of D can be labelled by elements of R such that

- at least two distinct elements of R are used
- at right-handed crossings, $f(x, y, z) = 0$
- at left-handed crossings, $g(x, y, z) = 0$

where x, y, z are the labels of the overstrand, incoming understrand, and outgoing understrand respectively.

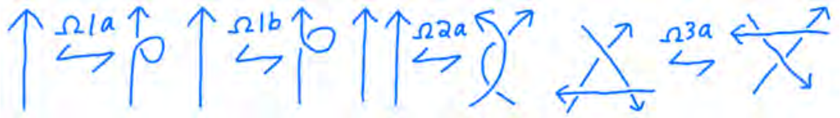


Reidemeister Moves for Oriented Knots

For oriented knots/links, there are 4 R1 moves, 4 R2 moves, and 8 R3 moves. Fortunately, they may be generated from a set of 4 moves:

Theorem

(Polyak 2010) Let D and D' be two diagrams of an oriented link L. Then D' can be obtained from D using isotopy and a finite sequence of the Reidemeister moves $\Omega 1a$*,* $\Omega 1b$*,* $\Omega 2a$*, and* $\Omega 3a$*. This set of 4 moves is also minimal.*



Sufficient Conditions for $(f, g)_R$ Colorability to be a Knot Invariant

Definition

We call a pair $(f, g)_R$ **strong** if the following properties are satisfied:

- Property $\Omega 1 a_1$: $\forall \alpha \in R,\ f(\alpha, \alpha, \alpha) = 0$
- Property $\Omega 1 a_2$: $\forall \alpha, \beta \in R,\ f(\alpha, \alpha, \beta) = 0 \implies \alpha = \beta$
- Property $\Omega 1 b_1$: $\forall \alpha, \beta \in R,\ f(\alpha, \beta, \alpha) = 0 \implies \alpha = \beta$
- Property $\Omega 2 a_1$: $\forall \alpha, \beta \in R, \exists \gamma \in R$ such that $f(\alpha, \beta, \gamma) = g(\alpha, \gamma, \beta) = 0$
- Property $\Omega 2 a_2$: $\forall \alpha, \beta, \gamma, \delta \in R$, $f(\alpha, \gamma, \delta) = g(\alpha, \delta, \beta) = 0 \implies \beta = \gamma$
- Property $\Omega 3 a$: $\forall \alpha, \beta, \gamma, \epsilon, \mu \in R$ such that $f(\alpha, \mu, \beta) = 0$, $\exists \delta \in R$ such that $f(\mu, \delta, \epsilon) = g(\alpha, \gamma, \delta) = 0$ if and only if $\exists \tau \in R$ such that $f(\beta, \gamma, \tau) = g(\alpha, \tau, \epsilon) = 0$

Sufficient Conditions for $(f,g)_R$ Colorability to be a Knot Invariant

Lemma

If $(f,g)_R$ is strong, then $(g,f)_R$ satisfies properties $\Omega 1a_1$, $\Omega 1a_2$, $\Omega 1b_1$, $\Omega 2a_1$, and $\Omega 2a_2$ (it turns out that $(g,f)_R$ also satisfies $\Omega 3a$, so $(g,f)_R$ is actually strong, but this requires a bit more work and is not needed for the next theorem).

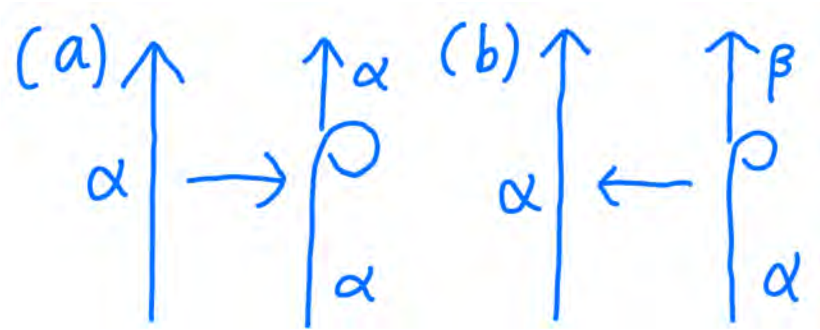
Theorem

If $(f,g)_R$ is strong, then $(f,g)_R$ colorability is a knot invariant.

Proof.

Covered in the next slides. □

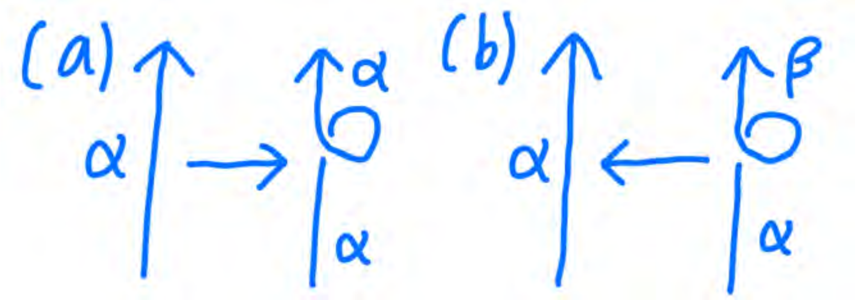
Invariance Under $\Omega 1a$ Reidemeister Move



(a) By Property $\Omega 1a_1$ for (f, g), $\forall \alpha \in R,\ f(\alpha, \alpha, \alpha) = 0$.

(b) By Property $\Omega 1a_2$ for (f, g),
$\forall \alpha, \beta \in R,\ f(\alpha, \alpha, \beta) = 0 \implies \alpha = \beta$.

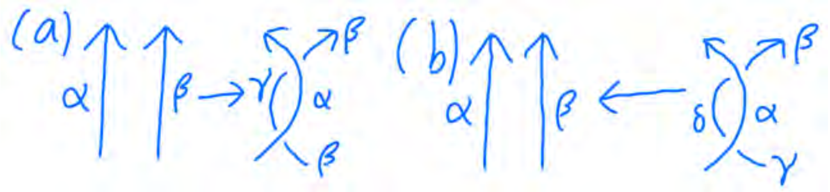
Invariance Under $\Omega 1b$ Reidemeister Move



(a) By Property $\Omega 1a_1$ for (g, f), $\forall \alpha \in R,\ g(\alpha, \alpha, \alpha) = 0$.

(b) By Property $\Omega 1b_1$ for (g, f),
$\forall \alpha, \beta \in R,\ g(\beta, \alpha, \beta) = 0 \implies \alpha = \beta$.

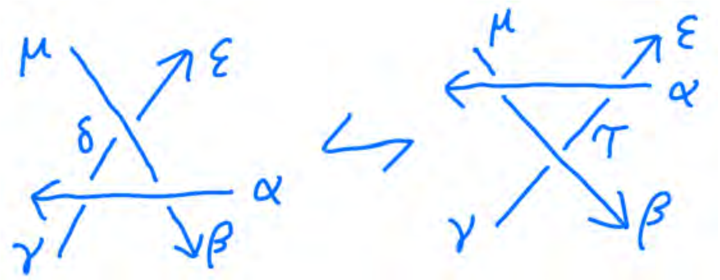
Invariance Under Ω2*a* Reidemeister Move



(a) By Property $\Omega 2a_1$ for (f, g), $\forall \alpha, \beta \in R, \exists \gamma \in R$ such that $f(\alpha, \beta, \gamma) = g(\alpha, \gamma, \beta) = 0$.

(b) By Property $\Omega 2a_2$ for (f, g), $\forall \alpha, \beta, \gamma, \delta \in R$, $f(\alpha, \gamma, \delta) = g(\alpha, \delta, \beta) = 0 \implies \beta = \gamma$. Note that if $\alpha = \beta$, then $f(\alpha, \gamma, \delta) = f(\alpha, \alpha, \delta) = 0$, so by Property $\Omega 1a_2$ for (f, g), $\delta = \alpha$. Hence (b) does not lead to a monochromatic coloring.

Invariance Under $\Omega 3a$ Reidemeister Move



By Property $\Omega 3a$ for (f, g), $\forall \alpha, \beta, \gamma, \epsilon, \mu \in R$ such that $f(\alpha, \mu, \beta) = 0$, $\exists \delta \in R$ such that $f(\mu, \delta, \epsilon) = g(\alpha, \gamma, \delta) = 0$ if and only if $\exists \tau \in R$ such that $f(\beta, \gamma, \tau) = g(\alpha, \tau, \epsilon) = 0$. Note that in either diagram, if all the colors except δ (resp. τ) were the same, then δ (resp. τ) would also be that same color by Properties $\Omega 1a_2$ and $\Omega 1b_1$.

Classification of Linear Strong Pairs Over Fields

Theorem

Let $\mathbb{F}$ be a field and $f, g \in \mathbb{F}[x, y, z]$ be linear. Then (f, g) is strong if and only if $f(x, y, z) = ax + by - (a + b)z$ and $g(x, y, z) = cf(x, z, y)$ for some $a, b, c \in \mathbb{F}$ such that $a + b, b, c \neq 0$.

Corollary

If $f \in \mathbb{F}[x, y, z]$ is linear and (f, f) is strong, then f is a multiple of the Fox n-coloring equation $2x - y - z$ or a multiple of $y - z$.

A Result for Quadratics in Strong Pairs over Fields

Theorem

Let $\mathbb{F}$ be a field with characteristic not equal to 2. Then if $f, g \in \mathbb{F}[x, y, z]$ and (f, g) is strong, then neither f nor g are irreducible quadratics.

Example of (f, g) Coloring

Above is an $(f, f)_{\mathbb{Z}/5\mathbb{Z}}$ coloring of the knot 4_1, where
$f(x, y, z) = 2xy + 2xz + 4yz + 3x^2 + 2y^2 + 2z^2$.

Quandles

Definition

A **quandle** is a set X equipped with a binary operation $*$ such that

(1) $\forall a \in X,\ a * a = a$

(2) $\forall b, c \in X,\ \exists!\, a \in X$ such that $a * b = c$

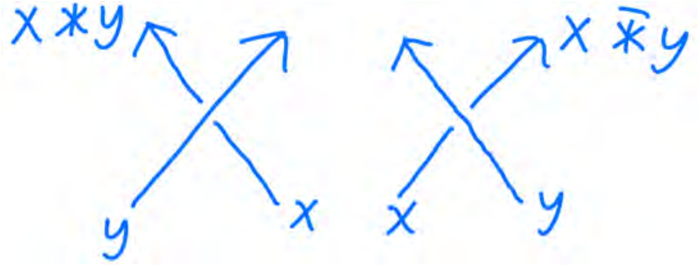
(3) $\forall a, b, c \in X,\ (a * b) * c = (a * c) * (b * c)$

Note that (2) is equivalent to the statement that for all $a \in X$, the transformation $R_a(x) := x * a$ for all $x \in X$ is invertible. We define the operation $\bar{*}$ on X by $a \,\bar{*}\, b = R_b^{-1}(a)$.

Coloring by a Quandle

Definition

A coloring of an oriented knot diagram K by a quandle $(X, *)$ is an assignment of an element of X to each arc of K such that at each crossing, the below relation holds. Such a coloring is called trivial if each arc is assigned the same element.



Quandle Colorability is a Knot Invariant

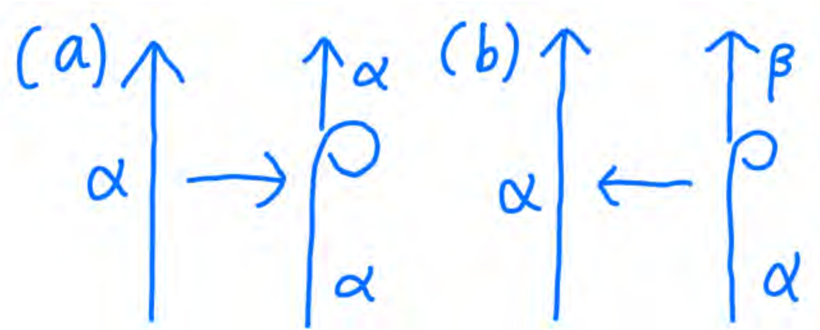
Theorem

Let $(X, *)$ be a quandle. The existence of a nontrivial $(X, *)$ coloring is a knot invariant. Furthermore, if X is finite, then the number of colorings by $(X, *)$ is a knot invariant.

Proof.

Covered in the next slides. □

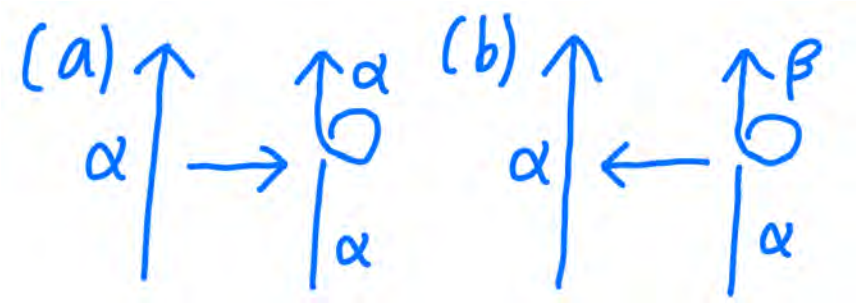
Invariance Under $\Omega 1a$ Reidemeister Move



(a) $\alpha * \alpha = \alpha$.

(b) $\beta = \alpha * \alpha = \alpha$.

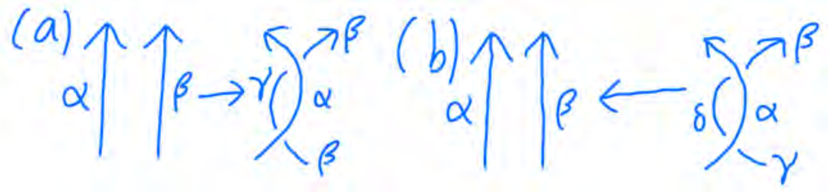
Invariance Under $\Omega 1b$ Reidemeister Move



(a) $\alpha * \alpha = \alpha$, so $\alpha = \alpha \,\bar{*}\, \alpha$.

(b) $\beta = \alpha \,\bar{*}\, \beta$, so $\beta = \beta * \beta = \alpha$.

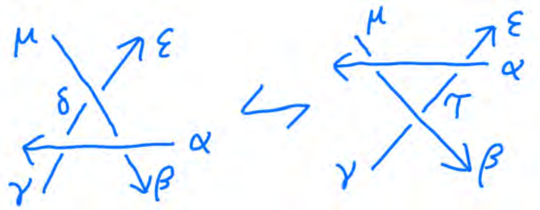
Invariance Under Ω2*a* Reidemeister Move



(a) Define $\gamma = \beta * \alpha$. Then $\beta = \gamma \,\bar{*}\, \alpha$.

(b) We have $\beta = \delta \,\bar{*}\, \alpha$ and $\delta = \gamma * \alpha$, so $\beta = (\gamma * \alpha) \,\bar{*}\, \alpha = \gamma$.

Invariance Under Ω3a Reidemeister Move



($\rightarrow$) We have $\epsilon = \delta * \mu$, $\beta = \mu * \alpha$, $\delta = \gamma \mathbin{\bar{*}} \alpha$. Define $\tau = \gamma * \beta$. Then $\epsilon * \alpha = (\delta * \alpha) * (\mu * \alpha) = \gamma * \beta = \tau$, so $\epsilon = \tau \mathbin{\bar{*}} \alpha$.

($\leftarrow$) We have $\epsilon = \tau \mathbin{\bar{*}} \alpha$, $\tau = \gamma * \beta$, $\beta = \mu * \alpha$. Define $\delta = \gamma \mathbin{\bar{*}} \alpha$. Then

$\epsilon * \alpha = \tau = \gamma * \beta = [(\gamma \mathbin{\bar{*}} \alpha) * \alpha] * [(\beta \mathbin{\bar{*}} \alpha) * \alpha] = [(\gamma \mathbin{\bar{*}} \alpha) * (\beta \mathbin{\bar{*}} \alpha)] * \alpha$,

so $\epsilon = (\gamma \mathbin{\bar{*}} \alpha) * (\beta \mathbin{\bar{*}} \alpha) = \delta * \mu$.

Linear Alexander Quandles

Definition

For $n, m \in \mathbb{Z}_+$ coprime, the linear Alexander quandle $LAQ(n, m)$ is the set $\mathbb{Z}/n\mathbb{Z}$ with the operation $a * b = ma + (1 - m)b \mod n$.

Proposition

$LAQ(n, m)$ is a quandle.

Note that when $n \geq 3$ and $m = -1$, $a * b = 2b - a$ mod n and $a \bar{*} b = 2b - a$ mod n, so $LAQ(n, -1)$ is exactly Fox n-coloring.

Relation Between LAQs and Strong (f, g)

Proposition

Every linear Alexander quandle $LAQ(n, m)$ forms a strong linear pair $(f, g)_{\mathbb{Z}/n\mathbb{Z}}$, where $f(x, y, z) = (1 - m)x + my - z$ and $g(x, y, z) = (1 - m^{-1})x + m^{-1}y - z$.

Proof.

We have $(1 - m) + m = 1$, so f is of the form $ax + by - (a + b)z$.
Also, $-m^{-1}f(x, z, y) = (1 - m^{-1})x - z + m^{-1}y = g(x, y, z)$.
Although the classification result for strong linear polynomials was stated for fields, a technical examination of its proof shows that for the reverse implication to hold, it is enough to work over a ring and require $a + b$, b, and c to have multiplicative inverses, where $g(x, y, z) = cf(x, z, y)$.
Alternatively, it is straightforward to check directly that the conditions for strongness hold. □

Example

Consider $LAQ(5,3)$. For all $x, y \in \mathbb{Z}/5\mathbb{Z}$, $y * x = 3y - 2x$ and if $z = y \,\bar{*}\, x$, then $y = z * x = 3z - 2x$, so $z = 2(2x + y) = 4x + 2y$. This gives the strong pair $(f, g)_{\mathbb{Z}/5\mathbb{Z}}$, where $f(x, y, z) = -2x + 3y - z$ and $g(x, y, z) = 4x + 2y - z$.

Further Questions

- Can $(f, g)_R$ colorability distinguish knots that Fox n-colorability can't?
- Is strongness a necessary condition for $(f, g)_R$ colorability to be a knot invariant?
- Can we construct quadratic or higher-order strong $(f, g)_R$ pairs from quandles?