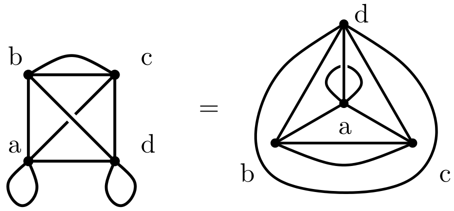
Graphs and their polynomials

Definition. A *graph* G is a finite set of *vertices* $V(G)$ and a finite set $E(G)$ of unordered pairs (x, y) of vertices $x, y \in V(G)$ called *edges.*

A graph may have *loops* (x, x) and *multiple edges* when a pair (x, y) appears in $E(G)$ several times. Pictorially we represent the vertices by points and edges by lines connecting the corresponding points. Topologically a graph is a 1-dimensional *cell complex* with $V(G)$ as the set of 0-cells and $E(G)$ as the set of 1-cells. Here are two pictures representing the same graph.



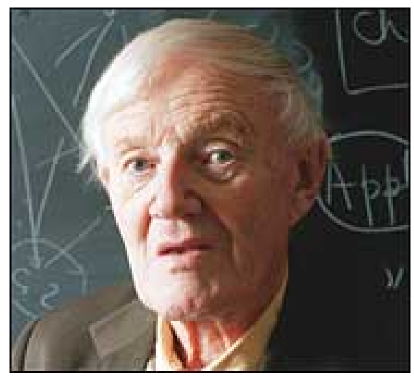
$V(G) = \{a, b, c, d\}$

$E(G) = \{(a, a), (a, b), (a, c), (a, d), (b, c), (b, c), (b, d), (c, d), (d, d)\}$

Chromatic polynomial $\chi_G(q)$.

A *coloring* of G with q colors is a map $\varkappa : V(G) \to \{1, \ldots, q\}$. A coloring $\varkappa$ is *proper* if for any edge e: $\varkappa(v_1) \neq \varkappa(v_2)$, where v_1 and v_2 are the endpoints of e.

Definition 1. *$\chi_G(q) :=$ # of proper colorings of G in q colors.*



Properties (Definition 2).

$\chi_G = \chi_{G-e} - \chi_{G/e}$;
$\chi_{G_1 \sqcup G_2} = \chi_{G_1} \cdot \chi_{G_2}$, for a disjoint union $G_1 \sqcup G_2$;
$\chi_\bullet = q$.

Tutte polynomial $T_G(x, y)$.

Definition 1.

$T_G = T_{G-e} + T_{G/e}$	*if e is neither a bridge nor a loop ;*
$T_G = xT_{G/e}$	*if e is a bridge ;*
$T_G = yT_{G-e}$	*if e is a loop ;*
$T_{G_1 \sqcup G_2} = T_{G_1 \cdot G_2} = T_{G_1} \cdot T_{G_2}$	*for a disjoint union $G_1 \sqcup G_2$ and a one-point join $G_1 \cdot G_2$;*

$T_\bullet = 1$.

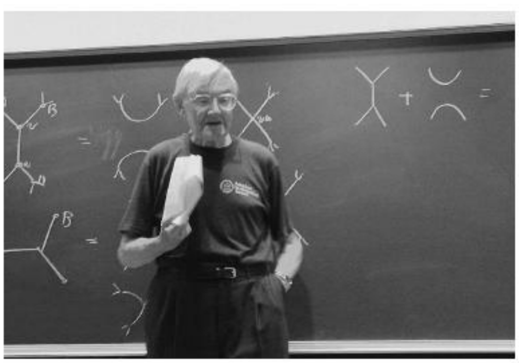


Fig. 1. W.T. Tutte. Photograph taken by Michel Las Vergnas at the CRM workshop, Barcelona, September 2001.
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Properties.

$T_G(1, 1)$	is the number of spanning trees of G ;
$T_G(2, 1)$	is the number of spanning forests of G ;
$T_G(1, 2)$	is the number of spanning connected subgraphs of G ;
$T_G(2, 2) = 2^{\lvert E(G)\rvert}$	is the number of spanning subgraphs of G ;

$\chi_G(q) = q^{k(G)}(-1)^{r(G)}T_G(1 - q, 0)$;

Definition 2. Let F be a graph;

- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of connected components of F;
- $r(F) := v(F) - k(F)$ be the *rank* of F;
- $n(F) := e(F) - r(F)$ be the *nullity* of F;

$$T_G(x, y) \;:=\; \sum_{F \subseteq E(G)} (x - 1)^{r(G) - r(F)} (y - 1)^{n(F)}$$

Dichromatic polynomial $Z_G(q,v)$ (Definition 3).

Let $Col(G)$ denote the set of colorings of G with q colors.

$$Z_G(q,v) \ := \sum_{\varkappa \in Col(G)} (1+v)^{\#\text{ edges colored not properly by } \varkappa}$$

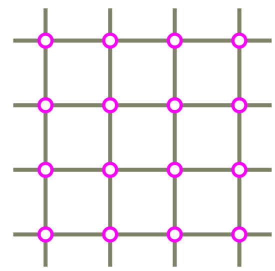
Properties .

$Z_G = Z_{G-e} + vZ_{G/e}$; $\quad Z_{G_1 \sqcup G_2} = Z_{G_1} \cdot Z_{G_2}$ for a disjoint union $G_1 \sqcup G_2$; $\quad Z_{\bullet} = q$;

$Z_G(q,v) = \sum_{F \subseteq E(G)} q^{k(F)} v^{e(F)}$; $\quad \chi_G(q) = Z_G(q,-1)$;

$Z_G(q,v) = q^{k(G)} v^{r(G)} T_G(1+qv^{-1}, 1+v)$; $\quad T_G(x,y) = (x-1)^{-k(G)}(y-1)^{-v(G)} Z_G((x-1)(y-1), y-1)$.

Potts model in statistical mechanics (Definition 4).



Potts model (C.Domb 1952); $\quad q=2$ the Ising model (W.Lenz, 1920)

Let G be a graph.

Particles are located at vertices of G. Each particle has a *spin*, which takes q different values . A *state, $\sigma \in \mathcal{S}$,* is an assignment of spins to all vertices of G. Neighboring particles interact with each other only if their spins are the same.

The energy of the interaction along an edge e is $-J_e$ (*coupling constant*). The model is called *ferromagnetic* if $J_e > 0$ and *antiferromagnetic* if $J_e < 0$.

Energy of a state σ (*Hamiltonian*),

$$H(\sigma) = - \sum_{(a,b)=e \in E(G)} J_e \ \delta(\sigma(a), \sigma(b)).$$

Boltzmann weight of σ:

$$e^{-\beta H(\sigma)} = \prod_{(a,b)=e \in E(G)} e^{J_e \beta \delta(\sigma(a),\sigma(b))} = \prod_{(a,b)=e \in E(G)} \Big(1 + (e^{J_e\beta} - 1)\delta(\sigma(a), \sigma(b))\Big),$$

where the *inverse temperature* $\beta = \frac{1}{\kappa\ T}$, T is the temperature, $\kappa = 1.38 \times 10^{-23}$ joules/Kelvin is the *Boltzmann constant*.

The Potts partition function (for $x_e := e^{J_e\beta} - 1$)

$$Z_G(q, x_e) \ := \sum_{\sigma \in \mathcal{S}} e^{-\beta H(\sigma)} = \sum_{\sigma \in \mathcal{S}} \prod_{e \in E(G)} (1 + x_e \delta(\sigma(a), \sigma(b)))$$

Properties of the Potts model Probability of a state σ: $\quad P(\sigma) := e^{-\beta H(\sigma)}/Z_G$.

Expected value of a function $f(\sigma)$:

$$\langle f \rangle := \sum_{\sigma} f(\sigma) P(\sigma) = \sum_{\sigma} f(\sigma) e^{-\beta H(\sigma)}/Z_G \ .$$

Expected energy: $\langle H \rangle = \sum_{\sigma} H(\sigma) e^{-\beta H(\sigma)}/Z_G = -\dfrac{d}{d\beta} \ln Z_G$.

Fortuin—Kasteleyn'1972: $\quad Z_G(q, x_e) = \sum_{F \subseteq E(G)} q^{k(F)} \prod_{e \in F} x_e$,

where $k(F)$ is the number of connected components of the spanning subgraph F.

$Z_G = Z_{G \setminus e} + x_e Z_{G/e}$.