Alternating knots and Tait Conjecture:

An application of Jones's Polynomial

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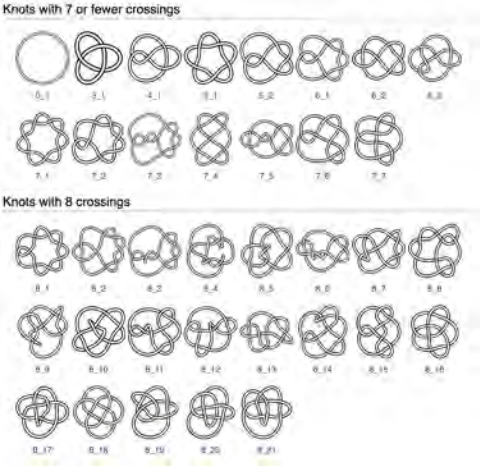
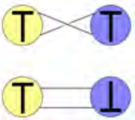
Outline



A knot is called **alternating knot** if it has an alternating projection diagram, which means:
As one travels along this knot, the crossings appears under and over alternatively.
Examples are listed on the left, all the knots before 8_{19} are alternating knots.
But how can you make sure some knots like 8_{19} is not isotopic to another alternating knot? Actually, this is not easy, but the Jones polynomial can answer it. We would not talk about it today.

When Tait made the atlas of knots in 19^{th} century, he noticed an experience law:
The diagram of an alternating knot tends to have the fewest crossings.
Let's do some observation:



We may see that the alternating property of upper diagram implies the same property of the lower diagram. By twisting the crossing in the center, we can reduce one crossing. We want to rule out such irregular cases.
We now state it mathematically the Tait conjecture:

Proposition:

Any reduced diagram of an alternating knot(link) has the fewest possible crossings.

Jones polynomial

A review

Denote $-A^3$ as α, $[\cdot]$ as Kauffman bracket, we have

$$f(L)(A) = \alpha^{-w(L)}[L] = g(A)$$

where $g \in \mathbb{Z}[x]$, $w(L)$ is the writhe of L.
The Jones polynomial $J(L)(t)$ is defined as $g(t^{-\frac{1}{4}})$, have the following several properties:

1) $t^{-1}J(L_+) - tJ(L_-) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})J(L_0)$
2) $J($  $) = 1$
3) Invariant under Reidemeister moves
4) $J(L_1 \# L_2) = J(L_1)J(L_2)$
5) $J(L_1 \sqcup L_2) = -(t^{\frac{1}{2}} - t^{-\frac{1}{2}})J(L_1)J(L_2)$

Coloring of classical diagram

Proposition:

A classical diagram can be colored by two numbers in a way that any two connected area have different colors.

Proof: By induction on the number of crossings. Here is the gist:

(Another proof: Since we know the 3-colorability, we also have a vector space isomorphism between 3-Fox coloring and 2-Dehn coloring.)

Lemma:

A connected diagram have n crossings, r regions, then $r = n + 2$

Proof: By induction.
(Another proof: By Euler's formula, $r - e + n = 2$, where e is the number of edges. $e = 2n$ because an edge determine 2 crossings, and each crossing has 4 edges.)

Preparation

All the diagrams are connected in the following discussion.

Definition: Span

Given a polynomial $f(x)$, *span* f is the highest degree of f minus the lowest degree of f.

Suppose L is a connected, reduced, checkerboard-colored alternating diagram.

$$[L] = \sum_{S} A^{i(S)-j(S)} d^{\delta(S)-1}$$

where $i(S), j(S), \delta(S)$ denote the number of A-splittings and B-splittings, and the number of circles in S, respectively, and $d = -A^2 - A^{-2}$

Guess: the highest term is from the state S that $j(S) = 0$. This term contributes $A^n d^{\delta(S)-1}$ to $[L]$, then the highest term is $\pm A^{n+2(\delta(S)-1)}$.

For example, given a diagram L, we color it.

We denote the number of black areas B, the number of white areas W.
Then, in the first picture, $n = 17$, $W = 7$, $B = 12$, and in the second picture, we can see $\delta(S) = 7 = W$.
We can let all black regions connected, and making this state that all the opened A-splittings coincide those black regions. Then this is an alternating diagram.
Hence, the "full-A state" has contribution:
$A^n d^{W-1} = A^n(-A^2 - A^{-2})^{W-1}$, whose highest term is $\pm A^{n+2(W-1)}$, likewise, we can guess the lowest term has contribution $\pm A^{-n-2(B-1)}$.
We will make this a theorem.

Theorem

The highest and lowest term of $[L]$ is $\pm A^{n+2(W-1)}$ and $\pm A^{-n-2(B-1)}$, hence $\mathrm{span}[L] = 4n$.

Proof: We analyze the highest term of $[L]$. Inheriting the notation from above, and we denote S', S'' as any states.
Notice that state S' would contribute: $A^{i(S')-j(S')}(-A^2 - A^{-2})^{\delta(S')-1}$, whose highest term has degree: $M(S') = i(S') - j(S') + 2(\delta(S') - 1)$.
(i). If state S' is obtained from S'' by changing one A-splitting to B-splitting, then $M(S') \leq M(S'')$, because $i(S') = i(S'') - 1$, $j(S') = j(S'') + 1$, $\delta(S') = \delta(S'') \pm 1$. If $\delta(S') = \delta(S'') + 1$, then $M(S') = M(S'')$; If $\delta(S') = \delta(S'') - 1$, then $M(S') = M(S'') - 4$.
(ii). If state S' is obtained from S by changing one A-splitting to B-splitting, then $\delta(S') = \delta(S) - 1$, thus $M(S') = M(S) - 4$
Thus we know that $\forall S'$, $n + 2(W - 1) = M(S) \geq M(S')$. Likewise, we know the lowest degree term is of the degree $-n - 2(B - 1)$.
Therefore, $\text{span}[L] = (n + 2(W - 1)) - (-n - 2(B - 1)) =$
$2n + 2(W + B - 2) = 2n + 2(r - 2) = 4n$
The last equality is by lemma: $r = n + 2$.
Hence,

The highest and lowest term of $[L]$ is $\pm A^{n+2(W-1)}$ and $\pm A^{-n-2(B-1)}$, hence $\text{span}[L] = 4n$

We can also see $\text{span}J(L)(t) = n$, which is an immediate result.

Lemma 1

$\delta(S) + \delta(\hat{S}) \leq n(L) + 2$, where $\hat{S}$ denotes the state that different from S totally.

Proof: Do induction on n. Since we assume the connectedness, we know when $n = 0$, $\delta(S) = \delta(\hat{S}) = 1$
Assume this proposition is true for $n < k$, then for $n = k$, choose any crossing P, split it without breaking the connectedness to get a $k - 1$ crossings diagram L'. (This is doable.) Either or not the crossing is A-type, we would have $\delta(S) + \delta(\hat{S}) \leq \delta(S') + \delta(\hat{S}') + 1$. Because only one of
(i). $\delta(S') = \delta(S'') \pm 1$, $\delta(\hat{S}') = \delta(\hat{S}'')$
(ii). $\delta(\hat{S}') = \delta(\hat{S}'') \pm 1$, $\delta(S') = \delta(S'')$
would happen.
Based on the inductive hypothesis, $\delta(S') + \delta(\hat{S}') \leq k - 1 + 2 = k + 1$, and therefore $\delta(S) + \delta(\hat{S}) \leq k + 1 + 1 = k + 2$.
Therefore, $\delta(S) + \delta(\hat{S}) \leq n(L) + 2$.

Theorem 2

For any L, $\text{span}J(L) \leq n(L)$.

Proof: Let S be the full-A-state. We see the highest term of $[L]$ has its degree $\leq n(L) + 2(\delta(S) - 1)$, while the lowest $\geq -n(L) - 2(\delta(\hat{S}) - 1)$.
Hence, $\text{span}[L] \leq 2n(L) + 2(\delta(S) + \delta(\hat{S}) - 2) \leq 4n(L)$
However, $J(L)(t) = n$.
And thus does $\text{span}J(L) \leq n(L)$.

Corollary

Any reduced diagram of an alternating link has the fewest possible crossings.

Proof: By Theorem 1, we know $\text{span}J(L)(t) = n(L)$, for reduced knot diagram.
By Theorem 2, $\text{span}J(L) \leq n(L)$.

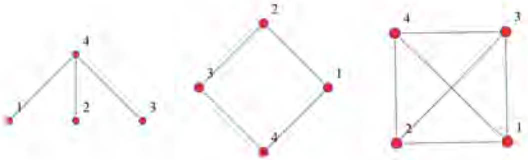
□

Disclaimer

Caution!

The following materials are at a primary stage,
and may contain fatal factual errors,
therefore might not be considered as a rigorous mathematical approach.

We can write a graph in the form of adjacent matrix, given the relations between any two points. We can also encode the information oriented graph into such structure, by deserting symmetry.



$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Basic Algebra

Equivalent Relation

Let A be a set, we can define the relation $\sim$ on $A \times A$, which satisfies the following:

1) $a \sim a$
2) If $a \sim b$, then $b \sim a$
3) If $a \sim b$ and $b \sim c$, then $a \sim c$

For example, $=$ is a relation on $\mathbb{R} \times \mathbb{R}$

What I want to do next is to encode the information of knot diagram into a matrix-like stuff, and encode the Reidemeister moves into equivalent relation.

Consider the free module $\mathcal{F}_{\mathbb{Z}_2}(P \times \mathbb{Z}_2 \times P \times \mathbb{Z}_2)$, where P is the set of crossings of a knot diagram. This is a $4|P|^2$-dimensional $\mathbb{Z}_2$ vector space.

As an analogue of the adjacent matrix, we can write the knot diagram as a table.

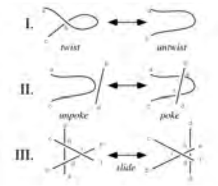
We can describe the edge between two crossings by listing two endpoints, as well as their positions.

For example, an edge between a and b can be written as (a, m_1, b, m_2), where $m_i \in \mathbb{Z}_2$ is used to describe the upper or lower position of the start and end of this edge.

And, the reason I choose to generate the module over $\mathbb{Z}/2\mathbb{Z}$ is because it can describe the binary relation of Yes and No. Also, $\mathbb{Z}/2\mathbb{Z}$ is a field, and $|P| < \infty$, so we actually generate a finite dimensional linear space.

Actually, the set of knot with $|P|$ crossings do not have a one-to-one correspondence with all the elements in this linear space. We need to put some restrictions, but it remains undone for the time being. But such construction is injective, an embedding into this linear space.

Given the Reidemeister moves, we should have:



R_1: $(a, m_1, b, m_2) + (b, m_2, b, \overline{m_2}) + (b, \overline{m_2}, c, m_3) - (a, m_1, c, m_3)$
R_2: $(a, m_1, e, m_2) + (e, m_2, f, m_2) + (f, m_2, c, m_3) + (b, m_4, e, \overline{m_2}) + (e, \overline{m_2}, f, \overline{m_2}) + (f, \overline{m_2}, d, m_5) - (a, m_1, c, m_3) - (b, m_4, d, m_5)$
R_3: $(a, m_1, g, m_2) + (g, m_2, h, m_2) + (h, m_2, d, m_3) + (b, m_4, g, \overline{m_2}) + (g, \overline{m_2}, i, m_5) + (i, m_5, e, m_6) + (c, m_7, h, \overline{m_2}) + (h, \overline{m_2}, i, \overline{m_5}) + (i, \overline{m_5}, f, m_8) - (b, m_4, i, m_5) - (i, m_5, h, m_2) - (h, m_2, e, m_6) - (c, m_7, i, \overline{m_5}) - (i, \overline{m_5}, g, m_2) - (g, m_2, f, m_8) - (a, m_1, g, m_2) - (g, m_2, h, m_2) - (h, m_2, d, m_3)$
$\forall a, b, c, d, e, f, g, h, i \in P, m_j \in \mathbb{Z}_2$
What I want is:

$$P \times \mathbb{Z}_2 \oslash P \times \mathbb{Z}_2 = \mathcal{F}_{\mathbb{Z}_2}(P \times \mathbb{Z}_2 \times P \times \mathbb{Z}_2)/(R_1, R_2, R_3)$$

Where (R_1, R_2, R_3) is the subspace generated by R_1, R_2, R_3

$$
\begin{array}{ccc}
P \times \mathbb{Z}_2 \times P \times \mathbb{Z}_2 & \longrightarrow & P \times \mathbb{Z}_2 \oslash P \times \mathbb{Z}_2 \\
& \searrow & \downarrow \\
& & N
\end{array}
$$

$\overline{\varphi} : P \times \mathbb{Z}_2 \times P \times \mathbb{Z}_2 \to N$ can be any knot invariant mapping.
Universal object is unique, but it is hard to prove the existence. But in the last page, if it is correct, I construct it explicitly.
Then the $P \times \mathbb{Z}_2 \oslash P \times \mathbb{Z}_2$ is the universal repelling object in the category of knot invariant mapping (N, φ) that makes the diagram commute.

THE END