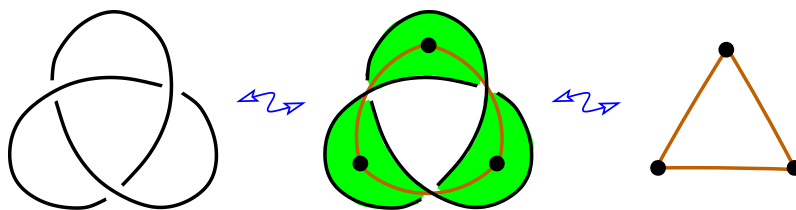


## Ribbon graphs

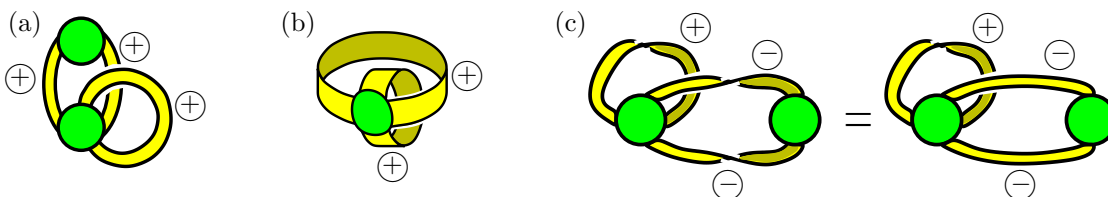
**Thistlethwaite's Theorem** [Ka1] *Up to a sign and multiplication by a power of  $t$  the Jones polynomial  $J_L(t)$  of an alternating link  $L$  is equal to the Tutte polynomial  $T_\Gamma(-t, -t^{-1})$ .*



The theorem was generalized to non-alternating links using signed graphs in [Ka2] and using the Bollobás-Riordan polynomial for ribbon graphs in [DFKLS]; to virtual links in [ChVo, Ch]; and to the arrow polynomial in [BBC].

**Definition.** A ribbon graph  $G$  is a surface (possibly non-orientable) with boundary, represented as the union of two sets of closed topological discs called *vertices*  $V(G)$  and *edges*  $E(G)$ , satisfying the following conditions:

- these vertices and edges intersect by disjoint line segments;
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.



## The Bollobás-Riordan polynomial

Reference: B. Bollobás and O. Riordan [BR].

$$R_G(\{x_e, y_e\}, X, Y, Z) := \sum_{F \subseteq G} \left( \prod_{e \in F} x_e \right) \left( \prod_{e \notin F} y_e \right) X^{r(G)-r(F)} Y^{n(F)} Z^{k(F)-bc(F)+n(F)}$$

For signed graphs, we set

$$\begin{cases} x_+ = 1, & x_- = (X/Y)^{1/2}, \\ y_+ = 1, & y_- = (Y/X)^{1/2}. \end{cases}$$

**Example.**

$(k, r, n, bc)$ term of $R_G$	$(1, 1, 1, 2)$ $X$	$(1, 1, 0, 1)$ $1$	$(1, 1, 0, 1)$ $1$	$(2, 0, 0, 2)$ $Y$
$R_G(X, Y, Z) = X + 2 + Y$ $+XYZ^2 + 2YZ + Y^2Z$				
	$(1, 1, 2, 1)$ $XYZ^2$	$(1, 1, 1, 1)$ $YZ$	$(1, 1, 1, 1)$ $YZ$	$(2, 0, 1, 2)$ $Y^2Z$

**Properties.**

$$R_G = x_e R_{G/e} + y_e R_{G-e}$$

$$R_G = (x_e + X y_e) R_{G/e}$$

$$R_{G_1 \sqcup G_2} = R_{G_1, G_2} = R_{G_1} \cdot R_{G_2}$$

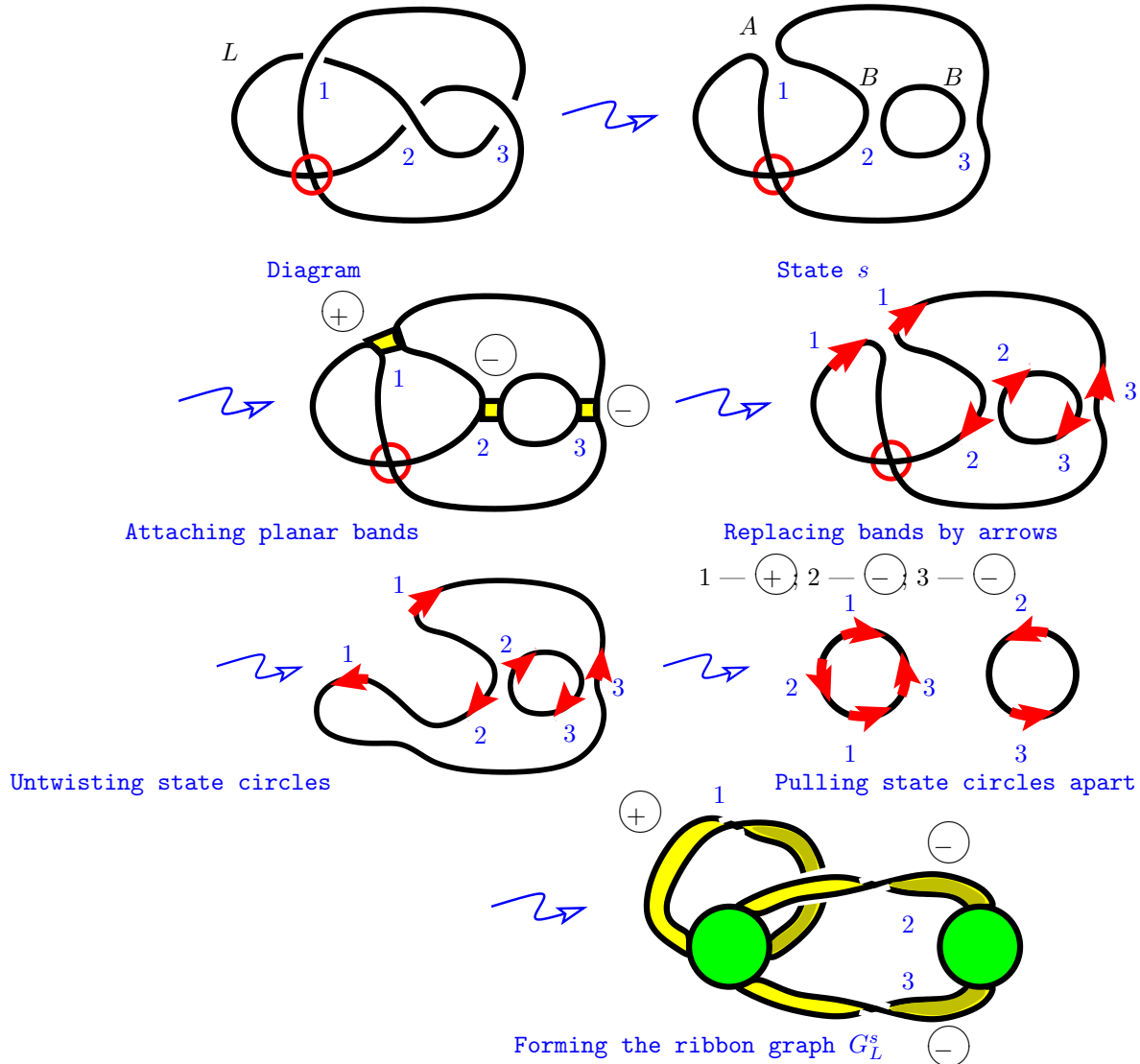
if  $e$  is ordinary, that is neither a bridge nor a loop,  
if  $e$  is a bridge.

**Theorem** [Ch].

Let  $L$  be a virtual link diagram with  $e$  classical crossings,  $G_L^s$  be the signed ribbon graph corresponding to a state  $s$ , and  $v := v(G_L^s)$ ,  $k := k(G_L^s)$ . Then  $e = e(G_L^s)$  and

$$[L](A, B, d) = A^e \left( X^k Y^v Z^{v+1} R_{G_L^s}(X, Y, Z) \Big|_{X=\frac{A^d}{B}, Y=\frac{B^d}{A}, Z=\frac{1}{d}} \right).$$

**Construction of a ribbon graph from a virtual link diagram**



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