Chromatic Polynomials of 2-Edge-Colored Graphs

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Chromatic Polynomials

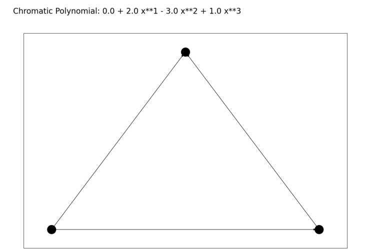
Definition

A *(proper) k-coloring* of a graph G is a function $d : V(G) \rightarrow \{1, 2, ..., k\}$ satisfying the following condition:

1. for all $yz \in E(G)$, we have $d(y) \neq d(z)$

Definition

The *chromatic polynomial* $P(G, k)$ is a polynomial that represents the number of (proper) k-colorings of G.

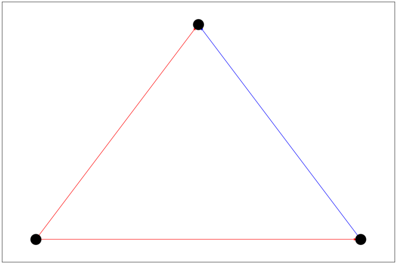


2-Edge-Colored Graphs

Definition

A *2-edge-colored graph* G is a triple (Γ, R_G, B_G), where Γ is a simple graph, $R_G \subseteq E(\Gamma)$, and $B_G \subseteq E(\Gamma)$ such that $R_G \cap B_G = \emptyset$ and $R_G \cup B_G = E(\Gamma)$.

Informally, it is a graph whose



k-coloring of a 2-Edge-Colored Graph

Definition

A *k-coloring* of a graph $G = (\Gamma, R, B)$ is a function $d : V(G) \to \{1, 2, ..., k\}$ satisfying the following two conditions:

1. for all $yz \in E(\Gamma)$, we have $d(y) \neq d(z)$; and
2. for all $ux \in R$ and $vy \in B$, if $d(u) = d(v)$, then $d(x) \neq d(y)$.

Definition

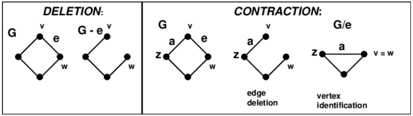
The *chromatic polynomial* $P(G, k)$ is a polynomial that represents the number of k-colorings of G.

How do we calculate chromatic polynomials?

Theorem

For a traditional graph G, we have the following contraction-deletion formula

$$P(G,k) = P(G-e,k) - P(G/e,k)$$



However, this no longer holds for 2-edge-colored graphs since property 2 is not local.

Mixed 2-Edge-Colored Graphs

It is convenient to introduce the following:

Definition

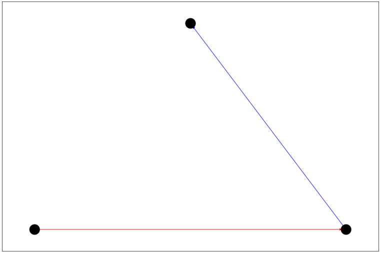
A *mixed 2-edge-colored graph* is a pair $M = (G, F_M)$ where G is a 2-edge-colored graph with $G = (\Gamma, R_G, B_G)$ and $F_M \subseteq E(\Gamma)$.

F_M can be considered as the set of edges that belong to neither R_G or B_G.

Definition

An *induced bichromatic 2-path* is an induced path uvy such that $uv \in R$ and $vy \in B$ or $uv \in B$ and $vy \in R$.

Induced Bichromatic 2-Path



Mixed 2-Edge-Colored Graphs

Lemma

If uvy is an induced bichromatic 2-path, then the vertices on the ends must be different colors.

Theorem

Let M be a mixed 2-edge-colored graph. If every pair of vertices in M are either adjacent in M or at the ends of an induced bichromatic 2-path in G, then in any coloring of M, each vertex receives a distinct color. Thus,

$$P(M, k) = \prod_{i=0}^{n-1} (k - i) = P(K_n, k)$$

Insertion-Contraction

Theorem

Let x and y be a pair of vertices that are neither adjacent in M nor at the ends of a bichromatic 2-path in M. Then,

$$P(M,k) = P(M+xy,k) + P(M_{xy},k)$$

where $M+xy$ is the mixed 2-edge-colored graph formed from M by adding xy to F and M_{xy} is the mixed 2-edge-colored graph formed from identifying vertices x and y and deleting any edge that is parallel with a colored edge.

Properties of Chromatic Polynomials

Theorem

If G has n vertices and c components, then the chromatic polynomial of G has the following properties:

1. *The coefficients of $k^0, k^1, ..., k^{c-1}$ are zero.*
2. *The coefficients of $k^c, ..., k^n$ are non-zero and alternate in sign.*
3. *The coefficient of k^n is 1.*

Corollary

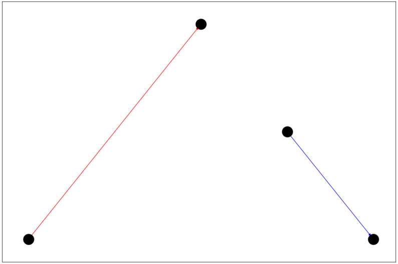
Since the coefficients are alternating in sign, $P(G, k)$ has no negative integer roots.

What about for 2-edge-colored graphs?

Typically, if G_1 and G_2 are disjoint, then

$$P(G_1 \cup G_2, k) = P(G_1, k) \cdot P(G_2, k)$$

However, in 2-edge-colored graphs, edges "far away" can affect each other.



Deletion-contraction and Whitney broken circuits for 2-edge colored graphs

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Deletion-contraction for 2-edge colored graphs

To get a proper deletion-contraction for 2-edge colored graphs we will generalize the idea of 2-edge colored graphs.

Definition

An *arbitrarily-edge-colored graph* G is a triple (Γ, F) where Γ is a simple graph, and $F \subset \{ab, |a, b \in E(\Gamma)\}$ such that $\forall e \in E(\Gamma)$, $ee \notin F$.

K-colorings of arbitrarily-edge-colored graphs

Definition

A *k-coloring* of $G = (\Gamma, F)$ is a function $d : V(G) \to \{1, 2, ..., k\}$ such that

1. for all $yz \in E(\Gamma)$, we have $d(y) \neq d(z)$; and
2. for all $ab \in F$ where $a = ux$ and $b = vy$, if $d(u) = d(v)$, then $d(x) \neq d(y)$.

Definition

The *chromatic polynomial* $P(G, k)$ is a polynomial that represents the number of (proper) k-colorings of G.

(The existence of this polynomial will follow directly from the deletion contraction bracket.)

Deletion contraction for arbitrarily-edge-colored graphs

Theorem

For an arbitrarily-edge-colored graph G we have the following deletion contraction formula.

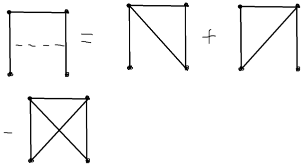
$\chi_G = \chi_{G-f} - \chi_{(G/f)_1} - \chi_{(G/f)_2}$.
Where, for $f = ab$ with $a = ux$ and $b = vy$, $u \neq v, y$, $x \neq v, x$, and $\chi_{(G/f)_1}$ represents the contractions of u onto v and x onto y, and $\chi_{(G/f)_2}$ represents the contractions of u onto y and x onto v.

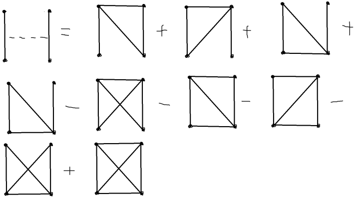
Proof.

The colorings of two contractions correspond exactly to the colorings of G with the edges a and b colored the same. So since $G - f$ has exactly these colorings in addition of the colorings of G, the theorem must follow. $\square$

For an arbitrarily-edge colored graph $G = (\Gamma, F)$ and for any $ab \in F$ with $a = ux$ and $b = uy$, consider the two pairs of edges $\{uv, xy\}$ and $\{uy, xv\}$. If and edge from each pair is contained in the edge set of Γ then both of the contractions of ab will contain a loop and so the chromatic polynomial of the deletion is equal to the chromatic polynomial of the original graph.
If at least one edge from one pair (but not from both) is contained in the edge set of G, assuming without loss of generality that $xy \in G$, then $\chi_G = \chi_{G-ab+xv} + \chi_{G-ab+uy} - \chi_{G-ab+xu+uy}$. This is because any k-coloring of $G - ab + xv$ or of $G - ab + uy$ is a k-coloring of G and all of the k-colorings of G are k-colorings of either $G - ab + xv$ or of $G - ab + uy$. Some of the colorings of G are both valid colorings of $G - ab + xv$ and of $G - ab + uy$ and these are exactly the k-colorings of $G - ab + xu + uy$.

If $uv, xy, uy, xv \notin E(G)$, then by the same argument
$\chi_{G-ab+xv+xy} + \chi_{G-ab+xv+uy} + \chi_{G-ab+uv+xy} + \chi_{G-ab+uv+uy} - \chi_{G-ab+xv+xy+uy} - \chi_{G-ab+xv+xy+uv} - \chi_{G-ab+xv+uy+uv} - \chi_{G-ab+xy+uv+uy} + \chi_{G-ab+xy+xy+uv+uy}$





Whitney's broken circuits of uncolored graphs

Definition

A broken circuit of a graph G is any subset of the edge set obtained by removing the maximal edge with respect to some linear ordering from some cycle which is a subgraph of G.

For any graph G, let the chromatic polynomial $P(G, \lambda) = \sum_{k=0}^{|V(G)|} (-1)^k a_k(G) \lambda^{|V(G)|-k}$

Theorem

(Whitney 1932) For any finite simple graphs G with some linear ordering of it's edges. Then for $k = 0, 1, ..., |V(G)|$, $a_k(G)$ equals the number of k-subsets of the edge set of G which do not include any broken circuits as subsets.

Whitney broken circuits for arbitrarily-edge-colored graphs

Theorem

For any arbitrarily-edge-colored graph $G = (\Gamma, F)$, consider the uncolored graph G' obtained by the following method:

1. *All vertices of G are contained in the vertex set of G'. All edges of G are contained in the edge set of G'.*
2. *For all $ab \in F$ with $a = ux$ and $b = vy$ such that $uv \in E(G)$ and $uy, vx \notin E(G)$, Add the pair of edges uy and vx to G'.*
3. *For all $ab \in F$ with $a = ux$ and $b = vy$ such that $uv, xy \notin E(G)$ and $uy, vx \notin E(G)$, Add the pair of edges uv and xy and the pair ux and vy to G'.*

Theorem

Consider any linear ordering on the edge-set of G' such that any edge from one of the pairs above receives an order less than that of any edge contain in the edge set of Γ. The coefficient $a_n(G)$ of the chromatic polynomial of G equal to the number of subsets of the edge set of G with no broken circuits as subsets and such that for any of the pairs of edges described above, either no edges are contained in this subset or both are and such that this subset contains an even number of pairs $-$ number of subsets of the edge set of G with no broken circuits as subsets and such that for any of the pairs of edges described above, either no edges are contained in this subset or both are and such that this subset contains an odd number of pairs.

Proof.

For any graph $G = (\Gamma, F)$, first, observe that this is true when F contains no elements. Then assume this theorem holds when F contains $n-1$ elements. Now, if F contains n elements, we will first consider the case of an $ab \in F$ corresponding to a pair of edges in G' added in step 2. We have that $\chi_{G-ab+xv} + \chi_{G-ab+uy} - \chi_{G-ab+xu+uy}$ and in this cases, all of the graphs $G - ab + xv$, $G - ab + uy$, and $G - ab + xu + uy$ have $n-1$ elements in F and so the theorem hold for these graphs. Let G' be the graph including all of $n-1$ the pairs not including ab. Consider any k-subset of $G' - ab + xv$ (or similarly of $G' - ab + uy$). If this subset does not contain the edge xv, then it is a subset of all three of these graphs and thus contributes 1 to the coefficient of k. f this subset does contain the edge xv, then it is also as subset of $G' - ab + xu + uy$ and so these cancel and contribute nothing to the coefficient. For any subset contained in $G' - ab + xu + uy$ but in neither $G' - ab + xv$ nor $G' - ab + uy$, this contributes -1 to the overall coefficient.

Proof.

Now consider the case of an $ab \in F$ corresponding to a pair of edges in G' added in step 2. We have that $\chi_{G-ab+xv+xy} + \chi_{G-ab+xv+uy} + \chi_{G-ab+uv+xy} + \chi_{G-ab+uv+uy} - \chi_{G-ab+xv+xy+uy} - \chi_{G-ab+xv+xy+uv} - \chi_{G-ab+xv+uy+uv} - \chi_{G-ab+xy+uv+uy} + \chi_{G-ab+xy+xy+uv+uy}$. Any k-subset containing just one of the edges xy is contained in $\chi_{G-ab+xv+xy}$, $\chi_{G-ab+uv+xy}$, $\chi_{G-ab+xv+xy+uy}$, $\chi_{G-ab+xv+xy+uv}$, $\chi_{G-ab+xy+uv+uy}$, and $\chi_{G-ab+xy+xy+uv+uy}$.

For a subset containing 2 of these edges, if they are not part of a pair of edges then the graph must contain a broken circuit. If they are then they are contained in 2 of the graphs with 3 added edges and one with four which contributes -1.

Any k-subset with 3 or four of these edges contains a broken circuit. .

$\square$

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