Chromatic Polynomials of 2-Edge-Colored Graphs

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2-Edge-Colored Graphs

Definition

A *2-edge-colored graph* G is a triple (Γ, R_G, B_G), where Γ is a simple graph, $R_G \subseteq E(\Gamma)$, and $B_G \subseteq E(\Gamma)$ such that $R_G \cap B_G = \emptyset$ and $R_G \cup B_G = E(\Gamma)$.

Definition

A *signed graph* is a graph whose edges are labelled with signs ($+$ or $-$).

Note that if $R_G, B_G \neq \emptyset$, we call the graph bichromatic and it can also be interpreted as a signed graph.

k-coloring of a 2-Edge-Colored Graph

Definition

A *k-coloring* of a graph $G = (\Gamma, R, B)$ is a function $d : V(G) \to \{1, 2, ..., k\}$ satisfying the following two conditions:

1. for all $yz \in E(\Gamma)$, we have $d(y) \neq d(z)$; and
2. for all $ux \in R$ and $vy \in B$, if $d(u) = d(v)$, then $d(x) \neq d(y)$.

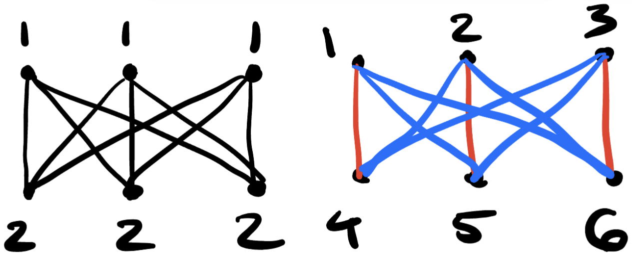
Definition

The *chromatic number* $\chi(G)$ is the least integer k such that G admits a k-coloring.

Definition

The *chromatic polynomial* $P(G, k)$ is a polynomial that represents the number of k-colorings of G.

Example



$$\chi(G_1) = 2, \chi(G_2) = 6$$

Mixed 2-Edge-Colored Graphs

Definition

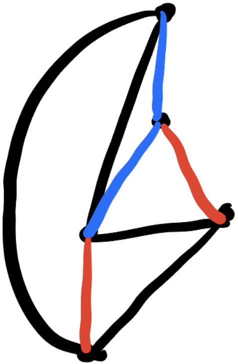
A *mixed 2-edge-colored graph* is a pair $M = (G, F_M)$ where G is a 2-edge-colored graph with $G = (\Gamma, R_G, B_G)$ and $F_M \subseteq E(\Gamma)$.

F_M can be considered as the set of edges that belong to neither R_G or B_G.

Definition

An *induced bichromatic 2-path* is an induced path uvy such that $uv \in R$ and $vy \in B$ or $uv \in B$ and $vy \in R$.

Example



Mixed 2-Edge-Colored Graphs

Lemma

If uvy is an induced bichromatic 2-path, then the vertices on the ends must be different colors.

Proof.

Assume that there exists a coloring of uvy. Suppose for a contradiction that $d(u) = d(y)$. Then, by property 2, v cannot be colored. Thus, $d(u) \neq d(y)$. □

Chromatic Polynomials of Mixed 2-Edge-Colored Graphs

Theorem

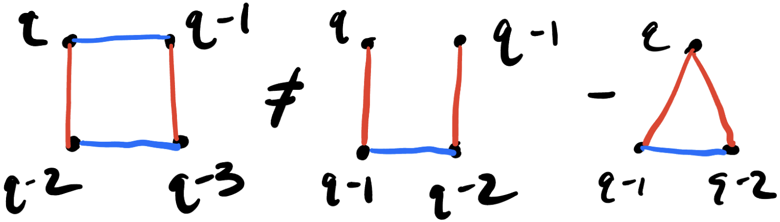
Let M be a mixed 2-edge-colored graph. If every pair of vertices in M are either adjacent in M or at the ends of an induced bichromatic 2-path in G, then in any coloring of M, each vertex receives a distinct color. Thus,

$$P(M, k) = \prod_{i=0}^{n-1} (k - i) = P(K_n, k)$$

Proof.

By previous lemma and property 1, every pair of vertices receive distinct colors. Thus, each vertex receives a distinct color. □

Example



Contraction-Deletion

The standard contraction-deletion formula

$$\chi_G = \chi_{G-e} - \chi_{G/e}$$

does not hold because property 2 is not local.

Insertion-Contraction

Theorem

Let x and y be a pair of vertices that are neither adjacent in M nor at the ends of a bichromatic 2-path in M. Then,

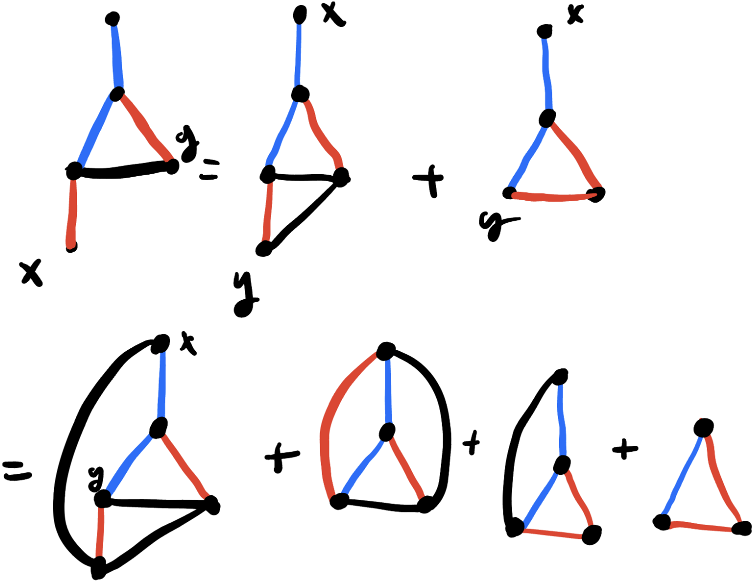
$$P(M, k) = P(M + xy, k) + P(M_{xy}, k)$$

where $M + xy$ is the mixed 2-edge-colored graph formed from M by adding xy to F and M_{xy} is the mixed 2-edge-colored graph formed from identifying vertices x and y and deleting any edge that is parallel with a colored edge.

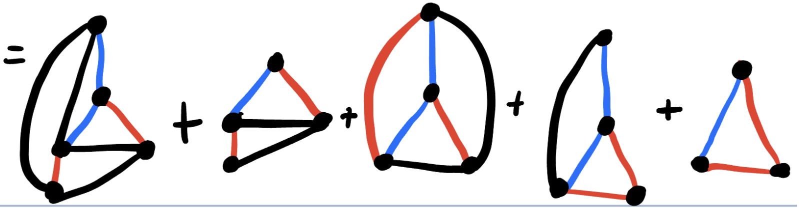
Proof.

The k-colorings of M can be partitioned into those in which x and y are the same color and those which they are different. □

Example



Example



$$= q(q-1)(q-2)(q-3)(q-4) + 3q(q-1)(q-2)(q-3) + q(q-1)(q-2)$$
$$= q^5 - 7q^4 + 18q^3 - 20q^2 + 8q$$

References

[BCDZ] I. Beaton, D. Cox, C. Duffy, N. Zolkavich, Chromatic Polynomials of 2-Edge-Coloured Graphs, The Electronic Journal of Combinatorics 30(4) (2023), #P4.40
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[Za] T. Zaslavsky, Signed graph coloring, Discrete Mathematics 39(2) (1982) 215–228.