Cut-vertex Recursion for the Intersection Polynomial

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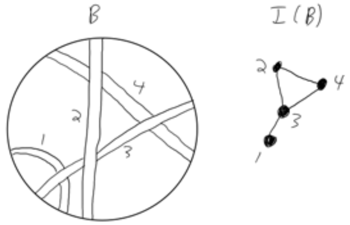
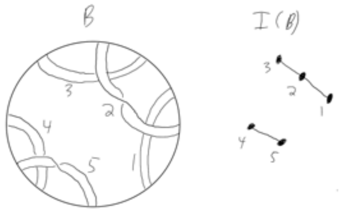
Intersection Graphs

Definition 1

The **intersection graph** of a bouquet B is the simple graph $\mathsf{I}(B)$ whose vertices are the edges of B such that two vertices of $\mathsf{I}(B)$ are adjacent if and only if the corresponding edges of B intersect.

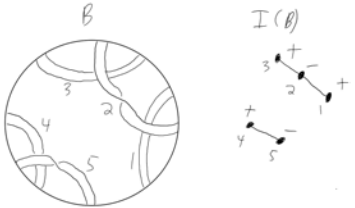




Signed Intersection Graphs

Definition 2

To record any half-twists of ribbon edges, we define the **signed intersection graph** of a bouquet B to be the signed graph $\mathrm{SI}(B)$ obtained by assigning a $+$ or $-$ to each vertex of $I(B)$ according to whether the corresponding ribbon edge is untwisted or twisted.



If all vertices of a signed intersection graph G are positive, then we call G positive.

Partial-dual Polynomial

Definition 3

Recall that the partial-dual Euler genus polynomial of a ribbon graph G is

$$^{\partial}\varepsilon_G(z) = \sum_{A \subseteq E(G)} z^{\varepsilon(G^A)},$$

the generating function enumerating the edge subsets of G by the Euler genus of their partial duals.

Intersection Polynomial

Theorem 4

(Yan and Jin 2022): If two bouquets B_1 and B_2 have the same signed intersection graph, then ${}^{\partial}\varepsilon_{B_1}(z) = {}^{\partial}\varepsilon_{B_2}(z)$.

Definition 5

Thus we can define the **intersection polynomial** $\mathrm{IP}_G(z)$ of a signed intersection graph G as the partial-dual Euler genus polynomial of any bouquet whose signed intersection graph is G.

A Recurrence Relation for Leaves

Theorem 6

(Yan and Jin 2022): Let G be a signed intersection graph and let v_1, v_2 be adjacent vertices of G such that v_1 is positive and has degree 1. Then

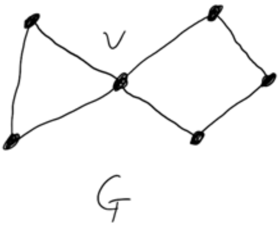
$$\mathrm{IP}_G(z) = \mathrm{IP}_{G-v_1}(z) + (2z^2)\,\mathrm{IP}_{G-v_1-v_2}(z).$$

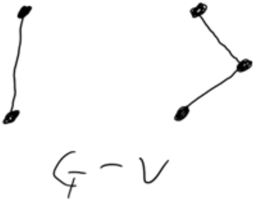
This allows us to recursively compute the intersection polynomial of any positive tree.

Cut-vertex

Definition 7

Let G be a graph. A vertex v of G is called a **cut-vertex** of G if $G - v$ has more connected components than G.





Cut-vertex Recurrence Relation

Theorem 8

Let G be a signed intersection graph such that $G = G_1 \cup G_2$, $G_1 \cap G_2 = \{v\}$, v is a cut-vertex of G, and $G_2 - v$ is positive. If v is positive then

$$\mathrm{IP}_G = \frac{2z^2 \mathrm{IP}_{G_1} \mathrm{IP}_{G_2 - v} + 2z^2 \mathrm{IP}_{G_1 - v} \mathrm{IP}_{G_2} - \mathrm{IP}_{G_1} \mathrm{IP}_{G_2} - 4z^2 \mathrm{IP}_{G_1 - v} \mathrm{IP}_{G_2 - v}}{2z^2 - 2}.$$

If v is negative then

$$\mathrm{IP}_G = \frac{2z^2 \mathrm{IP}_{G_1} \mathrm{IP}_{G_2 - v} + 2z^2 \mathrm{IP}_{G_1 - v} \mathrm{IP}_{G_2} - \mathrm{IP}_{G_1} \mathrm{IP}_{G_2} - 4z^3 \mathrm{IP}_{G_1 - v} \mathrm{IP}_{G_2 - v}}{2z^2 - 2z}.$$

Proof

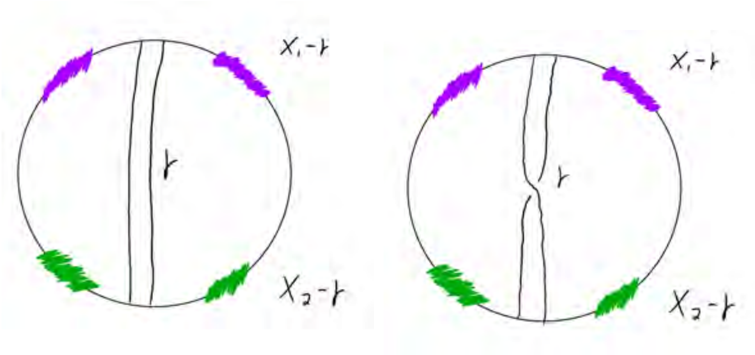
Let G_2 be a fixed signed intersection graph with a marked vertex v_2 such that $G_2 - v_2$ is positive. It suffices to show that for all signed intersection graphs G_1 with a marked vertex v_1, the graph G obtained by taking the disjoint union of G_1 and G_2 and identifying the vertices v_1 and v_2 satisfies the recursion, with $v = v_1 = v_2$.

First we show that there exist polynomials Q_1, Q_2 with non-negative integer coefficients such that

$$\mathrm{IP}_G = Q_1 \,\mathrm{IP}_{G_1} + Q_2 \,\mathrm{IP}_{G_1 - v}$$

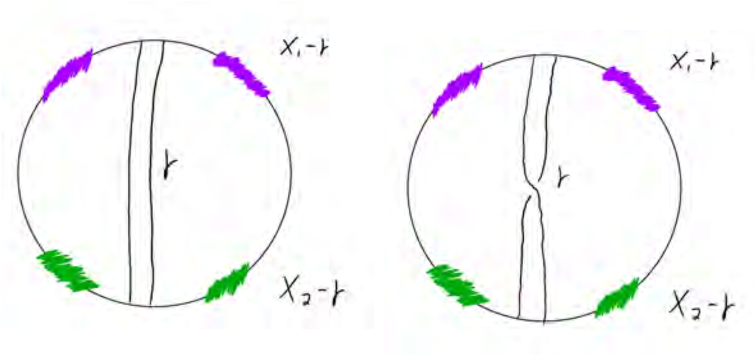
(note that Q_1, Q_2 depend on G_2 and v).

Proof



Let B be a bouquet with signed intersection graph G, let X_1, X_2 be the edge subsets corresponding to G_1, G_2, and let r be the edge corresponding to the cut-vertex v.

Proof



We can view B as above, where all edges in $X_1 - r$ have their ends in the purple regions and all edges in $X_2 - r$ have their ends in the green regions.

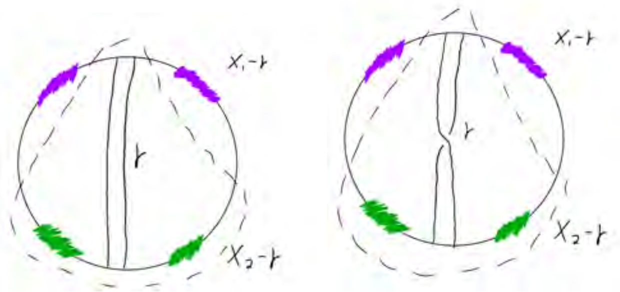
Proof

Consider the partition

$$\{A \subseteq E(B) : r \in A\} = \bigcup_{r \in F \subseteq X_2} T_F$$

where for $A \subseteq E(B)$ with $r \in A$, we have that $A \in T_F$ if and only if $F = A \cap X_2$. In other words, we sort the edge subsets of B containing r by their intersection with X_2.

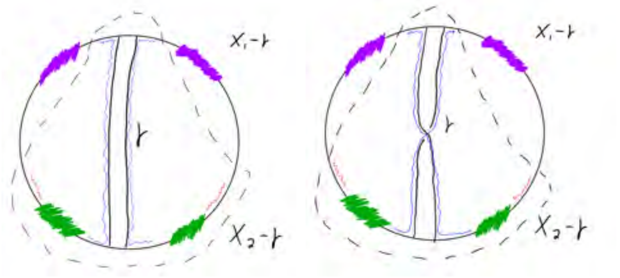
Proof



Now, for each such F, consider the region enclosed by the dashed lines in the ribbon subgraph of B whose edges in X_2 are precisely those in F and whose edges in $X_1 - r$ are arbitrary.
Since all the edges in $X_2 - r$ are untwisted, there are only two ways the boundary component starting in the top-left can exit.

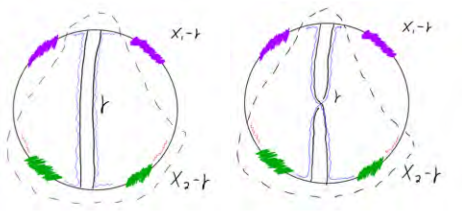
Proof

Case 1: Within the dashed region, the two sides of r belong to the same boundary component.



We have a one-to-one correspondence between subsets $D \subseteq E(X_1 - r)$ and $A \in T_F$ via $A = D \cup F$ (so $A^c = D^c \cup F^c$). For bouquets we have that $\varepsilon = 1 + e - f$, so $\varepsilon(A) = \varepsilon(D) + e(F) - b(F)$, where $e(F)$ is the number of edges in F and $b(F)$ is the number of boundary components lying completely inside the dashed region.

Proof



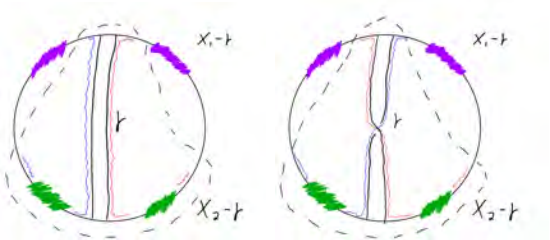
Similarly, $\varepsilon(A^c) = \varepsilon(D^c) + e(F^c) - b(F^c)$, so $\varepsilon(B^A) = \varepsilon(A) + \varepsilon(A^c) = \varepsilon((X_1 - r)^D) + \Delta(F)$ for some constant $\Delta(F) \in \mathbb{Z}$. Thus

$$\sum_F \sum_{A \in T_F} z^{\varepsilon(B^A)} = \sum_F z^{\Delta(F)} \sum_{D \subseteq E(X_1 - r)} z^{\varepsilon((X_1 - r)^D)} = \sum_F z^{\Delta(F)}\, {}^{\partial}\varepsilon_{X_1 - r}(z)$$

where the summation is taken over all $r \in F \subseteq X_2$ falling into case 1.

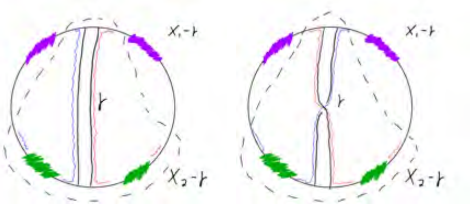
Proof

Case 2: Within the dashed region, the two sides of r belong to different boundary components.



We have a one-to-one correspondence between subsets $D \subseteq E(X_1)$ that contain r and $A \in T_F$ via $A = D \cup F$ (so $A^c = D^c \cup F^c$). Then once again, we have $\varepsilon(A) = \varepsilon(D) + e(F) - 1 - b(F)$, where $e(F)$ is the number of edges in F and $b(F)$ is the number of boundary components lying completely inside the dashed region.

Proof



Similarly, $\varepsilon(A^c) = \varepsilon(D^c) + e(F^c) - b(F^c)$, so
$\varepsilon(B^A) = \varepsilon(A) + \varepsilon(A^c) = \varepsilon(X_1^D) + \Delta(F)$ for some constant $\Delta(F) \in \mathbb{Z}$.
Thus

$$\sum_F \sum_{A \in T_F} z^{\varepsilon(B^A)} = \sum_F z^{\Delta(F)} \sum_{r \in D \subseteq E(X_1)} z^{\varepsilon(X_1^D)} = \frac{1}{2} \sum_F z^{\Delta(F)} \, {}^{\partial}\varepsilon_{X_1}(z)$$

where the summation is taken over all $r \in F \subseteq X_2$ falling into case 2.

Proof

Thus

$$\begin{aligned} {}^{\partial}\varepsilon_B(z) &= 2 \sum_{r \in A \subseteq E(B)} z^{\varepsilon(B^A)} \\ &= 2 \sum_{r \in F \subseteq X_2} \sum_{A \in T_F} z^{\varepsilon(B^A)} \\ &= Q_1\, {}^{\partial}\varepsilon_{X_1}(z) + Q_2\, {}^{\partial}\varepsilon_{X_1 - r}(z) \end{aligned}$$

for some $Q_1, Q_2 \in \mathbb{Z}[z]$, so

$$\mathrm{IP}_G(z) = Q_1\, \mathrm{IP}_{G_1}(z) + Q_2\, \mathrm{IP}_{G_1 - v}(z).$$

Proof

Given the existence of Q_1, Q_2, we can solve for them by plugging in specific values for G_1. First suppose v is positive. Setting $G_1 = \{v\}$, we get

$$\mathrm{IP}_{G_2} = \mathrm{IP}_G = Q_1 \mathrm{IP}_{G_1} + Q_2 \mathrm{IP}_{G_1 - v} = 2Q_1 + Q_2.$$

Now setting G_1 to be the graph with two positive vertices connected by an edge, we get that

$$\mathrm{IP}_G = Q_1 \mathrm{IP}_{G_1} + Q_2 \mathrm{IP}_{G_1 - v} = (2z^2 + 2)Q_1 + 2Q_2$$

and by Yan and Jin's recursion,

$$\mathrm{IP}_G = \mathrm{IP}_{G_2} + 2z^2 \mathrm{IP}_{G_2 - v}.$$

Proof

After solving the following system of equations,

$$\begin{cases} 2Q_1 + Q_2 = \mathrm{IP}_{G_2} \\ (2z^2+2)Q_1 + 2Q_2 = \mathrm{IP}_{G_2} + 2z^2\,\mathrm{IP}_{G_2-v} \end{cases}$$

we get

$$Q_1 = \frac{2z^2\,\mathrm{IP}_{G_2-v} - \mathrm{IP}_{G_2}}{2z^2-2},\ \ Q_2 = \mathrm{IP}_{G_2} - \frac{2z^2\,\mathrm{IP}_{G_2-v} - \mathrm{IP}_{G_2}}{z^2-1},$$

so

$$\mathrm{IP}_G = \frac{2z^2\,\mathrm{IP}_{G_1}\mathrm{IP}_{G_2-v} + 2z^2\,\mathrm{IP}_{G_1-v}\mathrm{IP}_{G_2} - \mathrm{IP}_{G_1}\mathrm{IP}_{G_2} - 4z^2\,\mathrm{IP}_{G_1-v}\mathrm{IP}_{G_2-v}}{2z^2-2}.$$

Proof

Similarly, if v is negative, then we set $G_1 = \{v\}$ to get

$$\mathrm{IP}_{G_2} = \mathrm{IP}_G = Q_1 \,\mathrm{IP}_{G_1} + Q_2 \,\mathrm{IP}_{G_1 - v} = (2z)Q_1 + Q_2.$$

If we now set G_1 to be the graph with one positive vertex connected to v, we get that

$$\mathrm{IP}_G = Q_1 \,\mathrm{IP}_{G_1} + Q_2 \,\mathrm{IP}_{G_1 - v} = (2z^2 + 2z)Q_1 + 2Q_2$$

and by Yan and Jin's recursion,

$$\mathrm{IP}_G = \mathrm{IP}_{G_2} + 2z^2 \,\mathrm{IP}_{G_2 - v}.$$

Proof

Solving the following system of equations

$$\begin{cases} (2z)Q_1 + Q_2 = \mathrm{IP}_{G_2} \\ (2z^2 + 2z)Q_1 + 2Q_2 = \mathrm{IP}_{G_2} + 2z^2\,\mathrm{IP}_{G_2 - v} \end{cases}$$

gives

$$Q_1 = \frac{2z^2\,\mathrm{IP}_{G_2-v} - \mathrm{IP}_{G_2}}{2z^2 - 2z}, \quad Q_2 = \mathrm{IP}_{G_2} - \frac{2z^2\,\mathrm{IP}_{G_2-v} - \mathrm{IP}_{G_2}}{z-1},$$

so

$$\mathrm{IP}_G = \frac{2z^2\,\mathrm{IP}_{G_1}\mathrm{IP}_{G_2-v} + 2z^2\,\mathrm{IP}_{G_1-v}\mathrm{IP}_{G_2} - \mathrm{IP}_{G_1}\mathrm{IP}_{G_2} - 4z^3\,\mathrm{IP}_{G_1-v}\mathrm{IP}_{G_2-v}}{2z^2 - 2z}.$$

Divisibility Check

We proved that the formulas

$$\mathrm{IP}_G = \frac{2z^2 \,\mathrm{IP}_{G_1} \mathrm{IP}_{G_2-v} + 2z^2 \,\mathrm{IP}_{G_1-v} \mathrm{IP}_{G_2} - \mathrm{IP}_{G_1} \mathrm{IP}_{G_2} - 4z^2 \,\mathrm{IP}_{G_1-v} \mathrm{IP}_{G_2-v}}{2z^2 - 2}$$

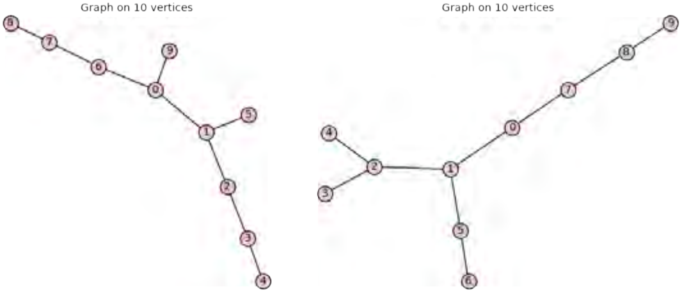
$$\mathrm{IP}_G = \frac{2z^2 \,\mathrm{IP}_{G_1} \mathrm{IP}_{G_2-v} + 2z^2 \,\mathrm{IP}_{G_1-v} \mathrm{IP}_{G_2} - \mathrm{IP}_{G_1} \mathrm{IP}_{G_2} - 4z^3 \,\mathrm{IP}_{G_1-v} \mathrm{IP}_{G_2-v}}{2z^2 - 2z}$$

are polynomials in z with integer coefficients, but is it apparent that the numerators of each are divisible by their denominators?

Yes, because:

- The intersection polynomial of any non-empty graph has even coefficients.
- If H is a signed intersection graph then $\mathrm{IP}_H(1) = 2^{V(H)}$, and if H is positive then IP_H is a polynomial in z^2, so $\mathrm{IP}_H(-1) = \mathrm{IP}_H(1)$.
- IP_H contains a non-zero constant term if and only if H is positive and bipartite.

Two Trees with the same Intersection Polynomial



The two trees above have the same intersection polynomial:

$$256z^8 + 480z^6 + 244z^4 + 42z^2 + 2.$$

So, the intersection polynomial does not distinguish trees.

References

Gross, Mansour, and Tucker, *Partial duality for ribbon graphs, I: Distributions* (2020)

Yan and Jin, *Partial-dual polynomials and signed intersection graphs* (2022)