Deletion Contraction and Whitney Broken Circuits for 2-Edge Colored Graphs

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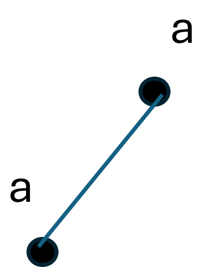
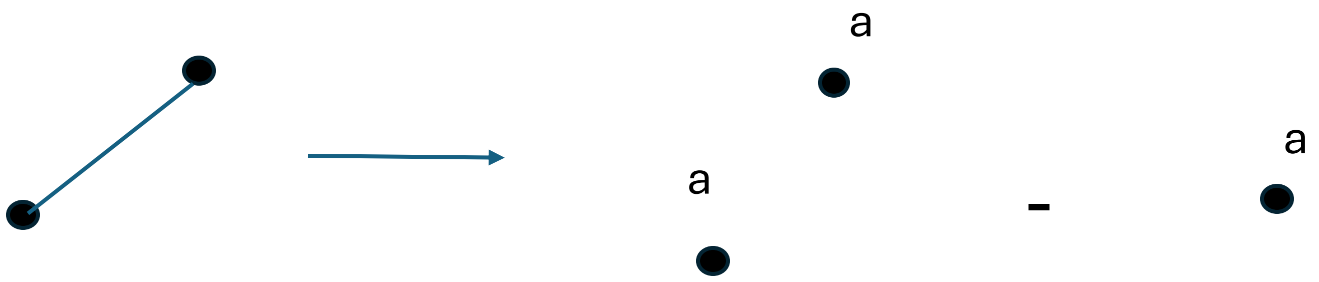
Let $M = (G, F)$ be a mixed 2-edge-coloured graph with $G = (\Gamma, R, B)$. We define a *k-colouring* of M to be a function $d : V(G) \to \{1, 2, 3, \ldots, k\}$ such that

1. $d(u) \neq d(v)$ for all $uv \in R \cup B \cup F$; and
2. for all $ux \in R$ and for all $vy \in B$, if $d(u) = d(v)$ then $d(x) \neq d(y)$.

Deletion contraction for unsigned graphs

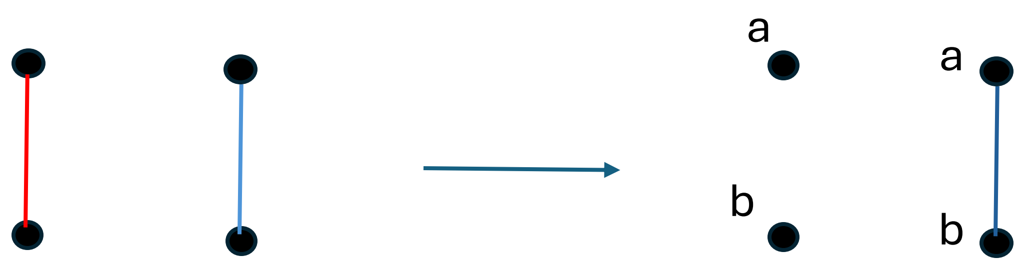
Forbidden coloring





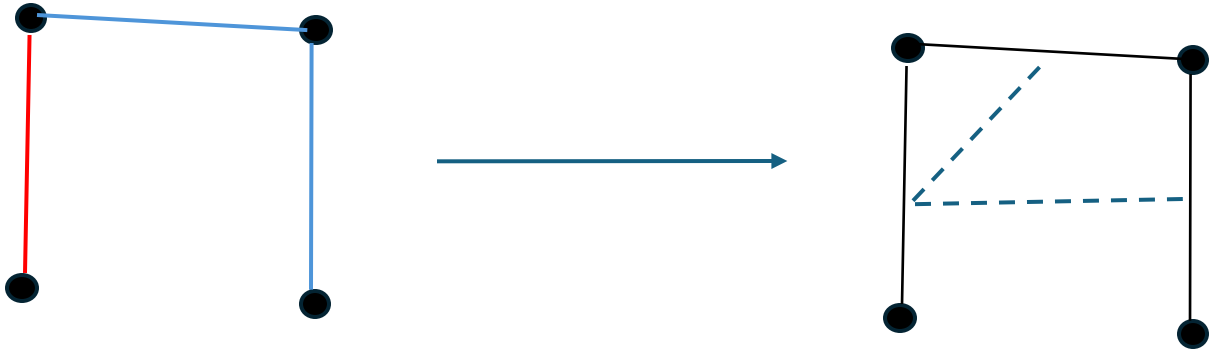
Deletion contraction for signed graphs

For signed graphs there are additional colorings which are forbidden in the original graph but allowed in the deletion

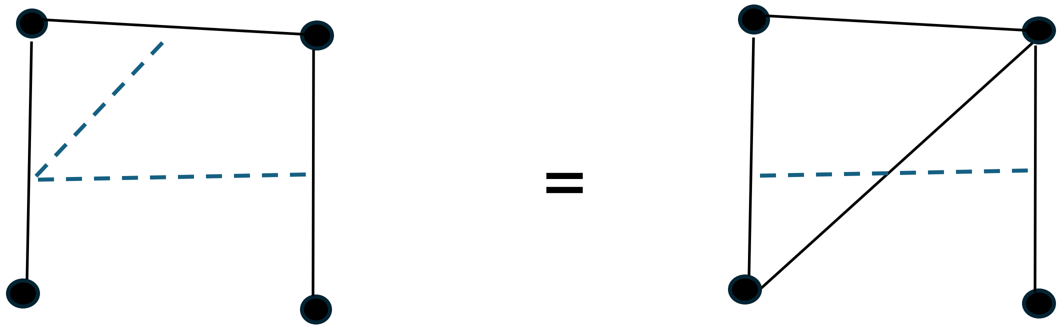


Modified notion of deletion contraction for signed graphs

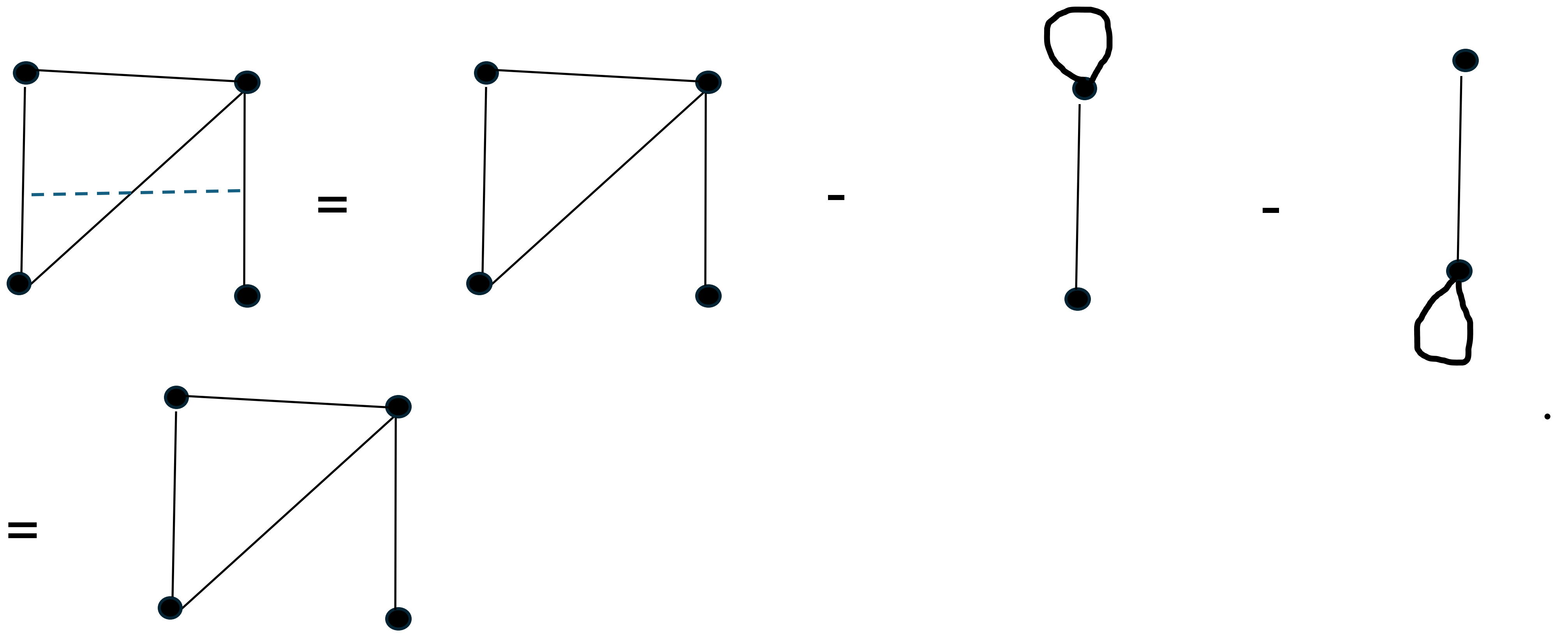
First, we will add additional "edges" between edges of opposite colors which tell us that both vertices of connected edges cannot share the same color



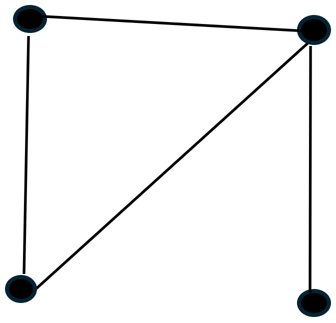
For any edges sharing a vertices these modified edges are the same as having an edge between the unconnected vertices



Now we will apply deletion contraction on these new "edges". To contract the "edges" we must consider both ways of identifying the vertices with each other.



Finally, we can apply normal deletion contraction



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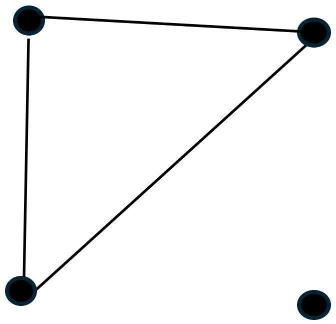

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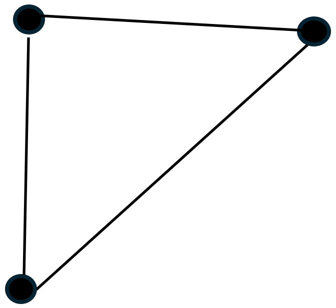


$$= \quad q(q-1)^2(q-2) \quad = \quad q^4 - 4q^3 + 5q^2 - 2q$$

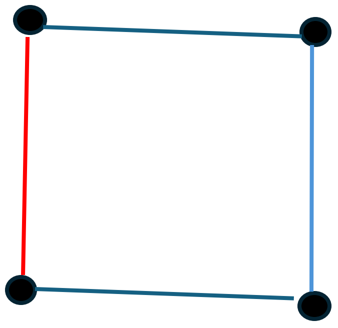
Whitney broken circuits for unsigned graphs

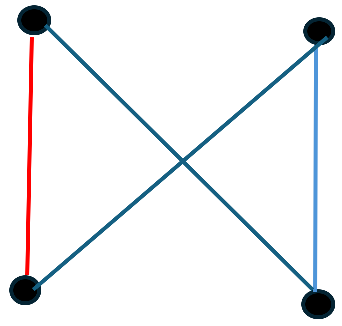
Definition: Let G be a graph with a linear ordering on its edge-set. We call a subset of G a broken circuit if it is a subset of a circuit of G obtained by deleting the maximal edge of that circuit with respect to the linear ordering.

Theorem (Whitney): Let G be a (finite simple) graph with a linear ordering on its edge-set. The coefficient of q^{n-k} in the chromatic polynomial of G (where n is the number of vertices of G) is equal to $(-1)^k$ times the the number of k-subsets of the edge-set of G without a broken circuit of G as a subset.

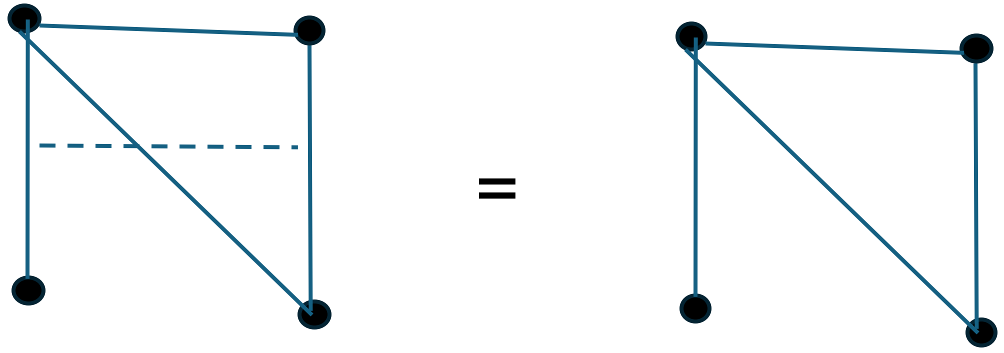
Whitney broken circuits for signed graphs

For each pair of edges of opposite signs we will consider the edges which could exist between their vertices as belonging to 2 sets which are the groups of 2 edges needed to create a four-cycle along with the signed edges.

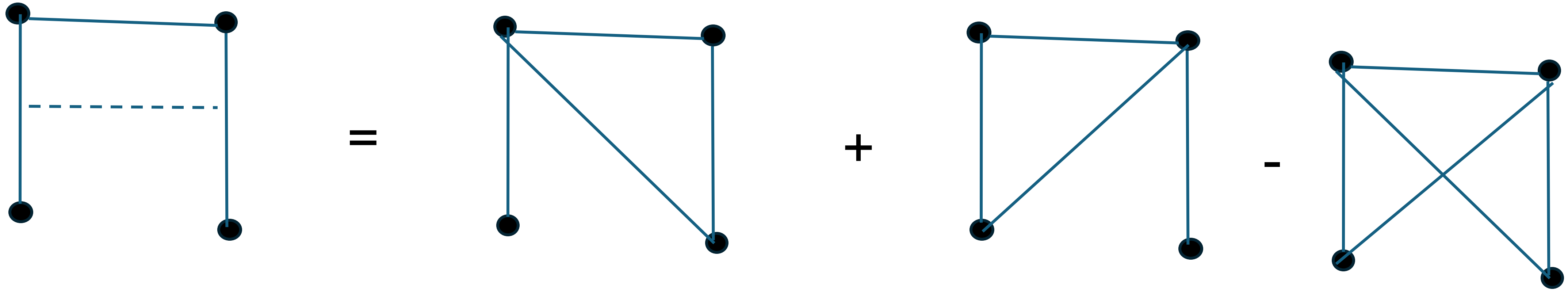




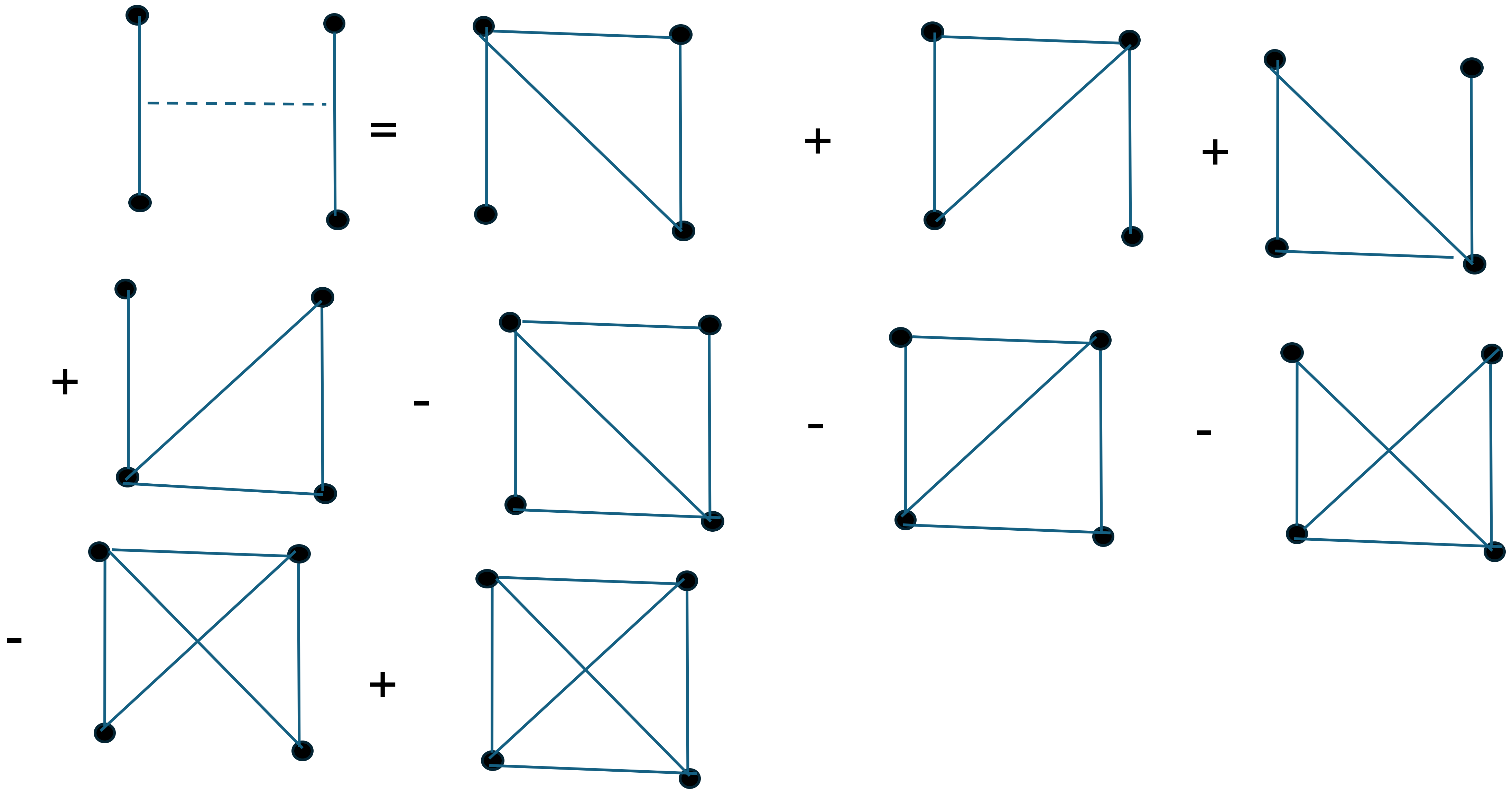
In any graph where at least one edge from both sets is present, we can remove the edge between these edges without changing the chromatic polynomial



If at least one edge from just one of these sets is present then we have

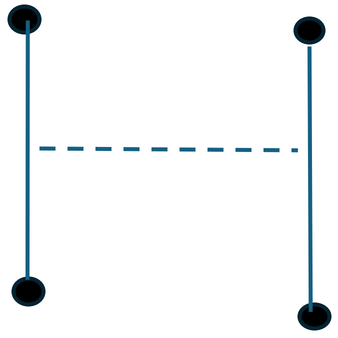


If neither edge from the set is in the graph we have



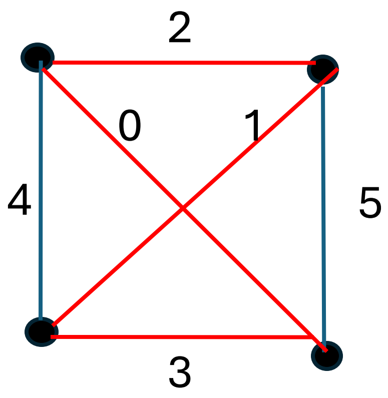
By looking at the Whitney circuits in this manner we get cancelation such that we only need to consider the cases where we have complete edge sets (unless we have a circuit made entirely from these edge sets)

Example of Whitney broken circuits for such a graph

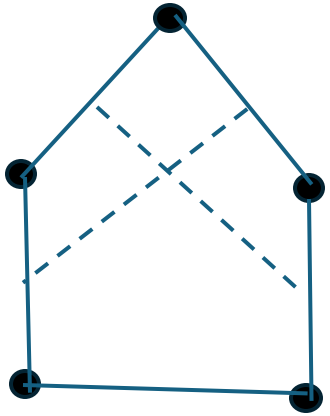


Chromatic polynomial of unsigned graph: $q^4 - 2q^3 + q^2$

Chromatic polynomial of signed graph: $q^4 - 2q^3 - q^2 + 2q$



Example 2



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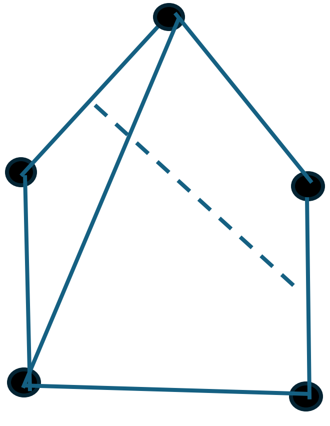

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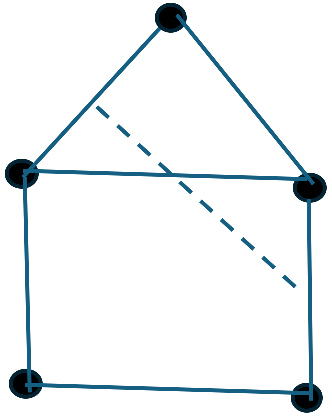

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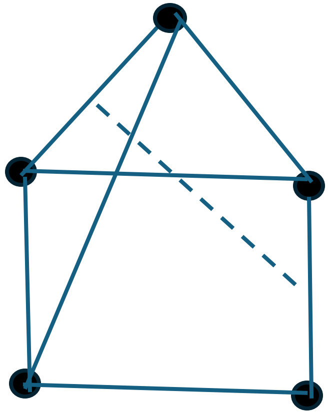

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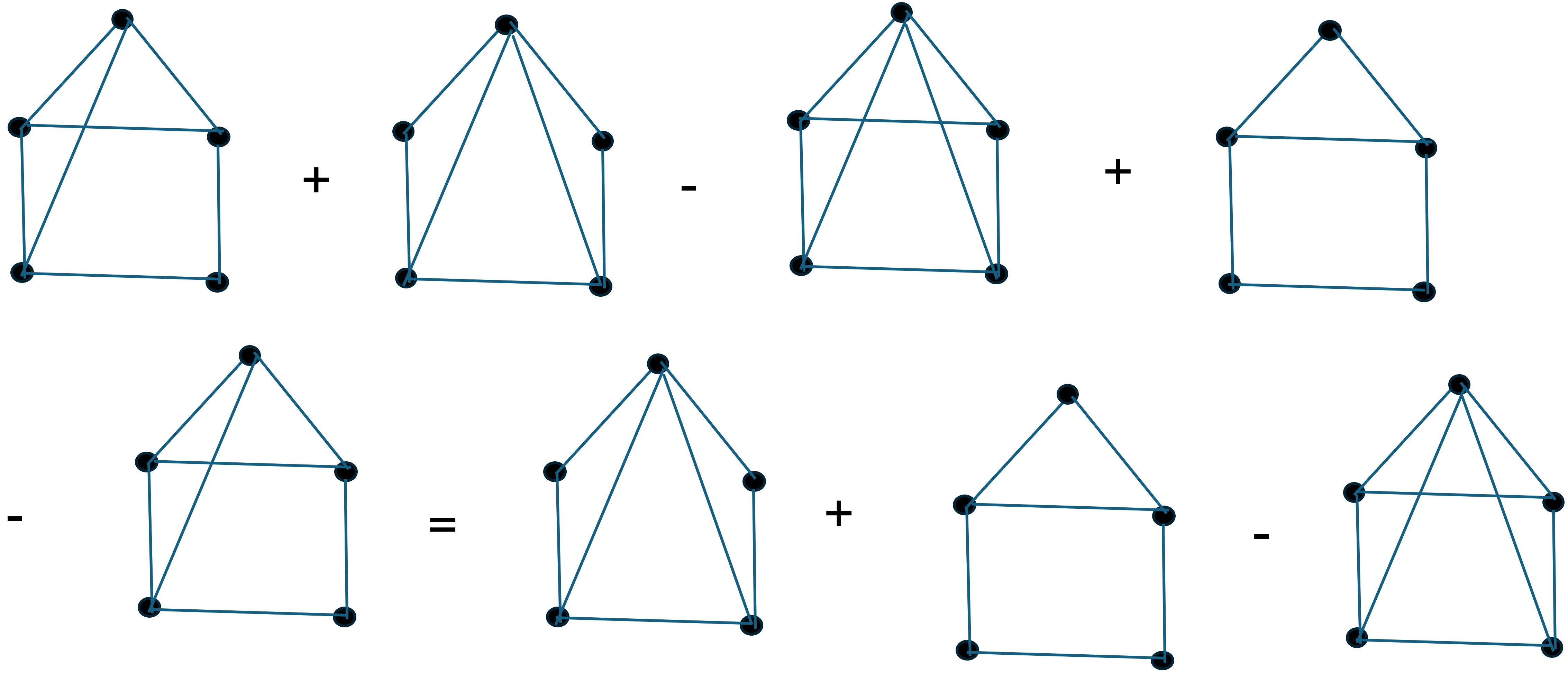
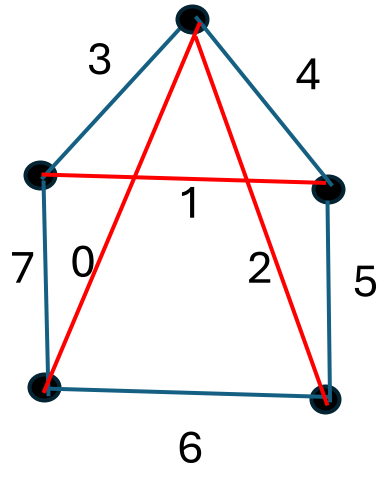
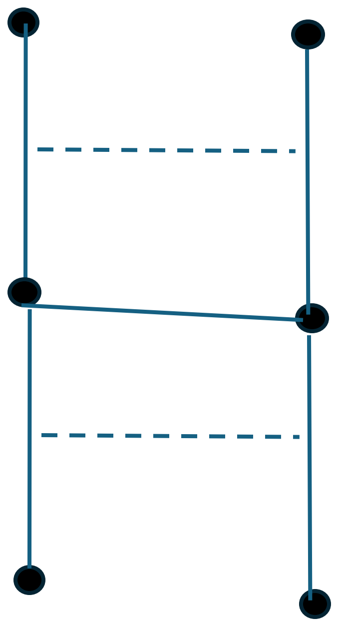


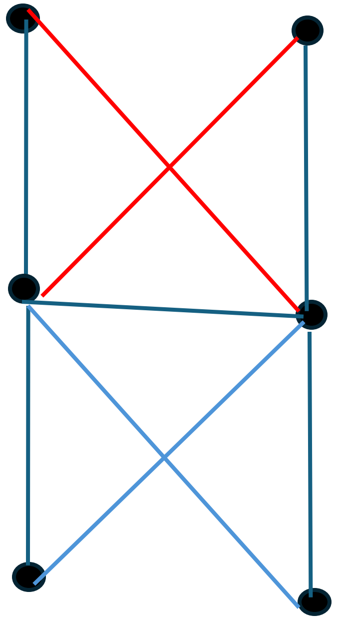


Chromatic polynomial of unsigned graph: $q^5 - 5q^3 + 10q^3 - 10q^2 + 4q$

Chromatic polynomial of signed graph: $q^5 - 5q^4 + 8q^2 - 4q^2$

Example 3





Chromatic polynomial of unsigned graph: $q^6 - 5q^5 + 10q^4 - 10q^3 + 5q^2 + q$

Chromatic polynomial of signed graph: $q^6 - 5q^5 + 8q^4 - 4q^3$

Sources

- I. Beaton, D. Cox, C. Duffy, N. Zolkavich, Chromatic Polynomials of 2-Edge-Coloured Graphs, The Electronic Journal of Combinatorics 30(4) (2023), #P4.40 https://doi.org/10.37236/9785
- Dohmen, K. An Inductive Proof of Whitney's Broken Circuit Theorem, Discussiones Mathematicae, Graph Theory 31 (2011) 509–515 https://doi.org/10.48550/arXiv.0912.1182