Intersection Graphs and Partial Duality

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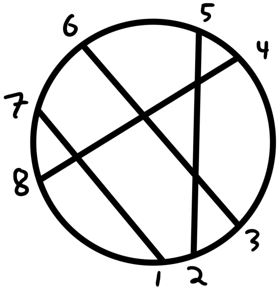
Chord Diagrams

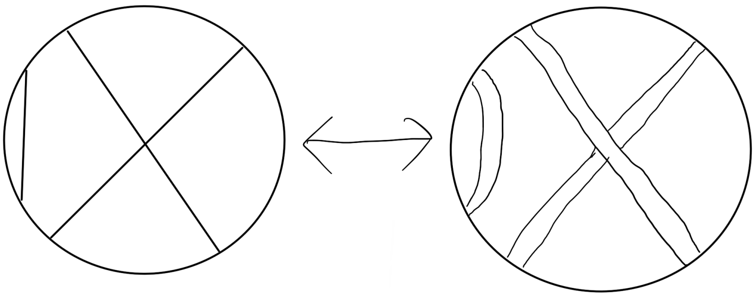
Definition 1

A **chord diagram** is a collection of chords on a circle.



Abstractly, a chord diagram is a cyclic ordering of objects that are partitioned into pairs.

Chord Diagrams as Ribbon Graphs

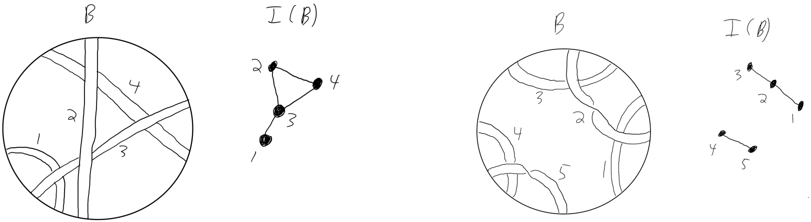


We can view any chord diagram as an orientable ribbon graph with one vertex (bouquet). Conversely, any orientable bouquet can be seen as a chord diagram.

Intersection Graphs

Definition 2

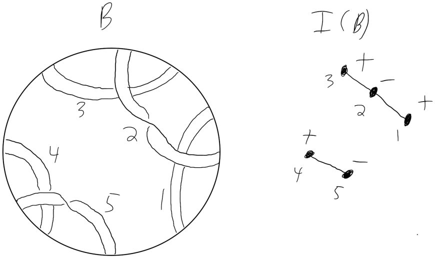
The **intersection graph** of a bouquet B is the simple graph $\mathrm{I}(B)$ whose vertices are the edges of B such that two vertices of $\mathrm{I}(B)$ are adjacent if and only if the corresponding edges of B intersect.



Signed Intersection Graphs

Definition 3

To record any half-twists of ribbon edges, we define the **signed intersection graph** of a bouquet B to be the signed graph $\mathrm{SI}(B)$ obtained by assigning a $+$ or $-$ to each vertex of $I(B)$ according to whether the corresponding ribbon edge is untwisted or twisted.



Partial-dual Polynomial

Definition 4

Recall that the partial-dual Euler genus polynomial of a ribbon graph G is

$$^{\partial}\varepsilon_G(z) = \sum_{A \subseteq E(G)} z^{\varepsilon(G^A)},$$

the generating function enumerating the edge subsets of G by the Euler genus of their partial duals.

Intersection Polynomial

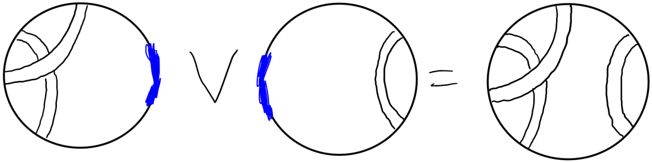
Theorem 5

(Yan and Jin 2022): If two bouquets B_1 and B_2 have the same signed intersection graph, then $^{\partial}\varepsilon_{B_1}(z) = {}^{\partial}\varepsilon_{B_2}(z)$.

Definition 6

Thus we can define the **intersection polynomial** $\mathrm{IP}_G(z)$ of a signed intersection graph G as the partial-dual Euler genus polynomial of any bouquet whose signed intersection graph is G.

Ribbon Join



Definition 7

Let R_1, R_2 be disjoint ribbon graphs. The **ribbon join** $R_1 \vee R_2$ is obtained by identifying a boundary arc on a vertex disc of R_1 with an arc on a vertex disc of R_2. This operation is in general not unique.

Proposition 1

$${}^{\partial}\varepsilon_{R_1 \vee R_2}(z) = {}^{\partial}\varepsilon_{R_1}(z)\,{}^{\partial}\varepsilon_{R_2}(z).$$

Disjoint Union

Proposition 2

Let B_1 and B_2 be bouquets. Then

$$\mathrm{SI}(B_1 \vee B_2) = \mathrm{SI}(B_1) \sqcup \mathrm{SI}(B_2).$$

Thus if G_1 and G_2 are signed intersection graphs, then

$$\mathrm{IP}_{G_1 \sqcup G_2}(z) = \mathrm{IP}_{G_1}(z)\,\mathrm{IP}_{G_2}(z).$$

A Recurrence Relation

Theorem 8

(Yan and Jin 2022): Let G be a signed intersection graph and let v_1, v_2 be adjacent vertices of G such that v_1 is positive and has degree 1. Then

$$\mathrm{IP}_G(z) = \mathrm{IP}_{G-v_1}(z) + (2z^2)\,\mathrm{IP}_{G-v_1-v_2}(z).$$

This allows us to recursively compute the intersection polynomial of any positive tree.

A Recurrence Relation

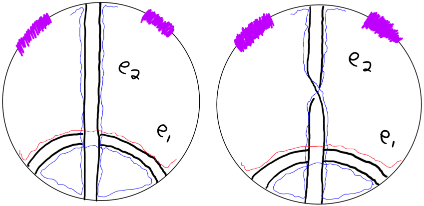
Proof.

Let B be a bouquet such that $\mathrm{SI}(B) = G$, and let e_1, e_2 be the ribbon edges corresponding to v_1, v_2. Partition the edge subsets of B into τ_1, τ_2, where for $A \subseteq E(B)$, $A \in \tau_1$ iff A contains one of e_1, e_2 but not the other, and $A \in \tau_2$ iff A contains both or neither of e_1, e_2.

Let $D \subseteq E(B - e_1)$. We consider D as a spanning subgraph of $B - e_1$, so that $D^c = E(B - e_1) - D$. If $e_2 \in D$ then take $A = D$, so that $A^c = D^c \cup e_1$. If $e_2 \notin D$ then take $A = D \cup e_1$, so that $A^c = D^c$. In both cases, $\varepsilon(A) = \varepsilon(D)$ and $\varepsilon(A^c) = \varepsilon(D^c)$ since the addition of an untwisted ribbon edge that doesn't intersect with any other edge preserves the Euler genus. Thus $\varepsilon(B^A) = \varepsilon((B - e_1)^D)$, so

$$\sum_{A \in \tau_1} z^{\varepsilon(B^A)} = {}^{\partial}\varepsilon_{B-e_1}(z).$$

A Recurrence Relation



Proof.

Now let $D \subseteq E(B - e_1 - e_2)$, considered as a spanning subgraph of $B - e_1 - e_2$, so that $D^c = E(B - e_1 - e_2) - D$. Let $A = D \cup \{e_1, e_2\}$, so that $A^c = D^c$. Note that A and D have the same number of faces, so $\varepsilon(A) = \varepsilon(D) + 2$ (recall that $\varepsilon = 1 + e - f$ for bouquets). Thus $\varepsilon(B^A) = \varepsilon((B - e_1 - e_2)^D) + 2$, so

$$\sum_{A \in \tau_2} z^{\varepsilon(B^A)} = 2 \sum_{\{e_1, e_2\} \subseteq A \in \tau_2} z^{\varepsilon(B^A)} = (2z^2)\, {}^{\partial}\varepsilon_{B - e_1 - e_2}(z).$$

A Recurrence Relation

Proof.

Therefore

$$\begin{aligned}\partial\varepsilon_B(z) &= \sum_{A\in\tau_1} z^{\varepsilon(B^A)} + \sum_{A\in\tau_2} z^{\varepsilon(B^A)} \\ &= \partial\varepsilon_{B-e_1}(z) + (2z^2)\,\partial\varepsilon_{B-e_1-e_2}(z),\end{aligned}$$

i.e.,

$$\mathrm{IP}_G(z) = \mathrm{IP}_{G-v_1}(z) + (2z^2)\,\mathrm{IP}_{G-v_1-v_2}(z).$$

□

Example (n-Paths)

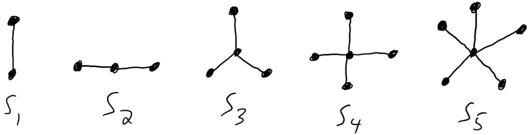
Let P_n be the path with n positive vertices. Then

$$\mathrm{IP}_{P_1}(z) = 2, \ \mathrm{IP}_{P_2}(z) = 2+2z^2, \ \mathrm{IP}_{P_{n+2}}(z) = \mathrm{IP}_{P_{n+1}}(z)+2z^2\,\mathrm{IP}_{P_n}(z).$$

For example, $\mathrm{IP}_{P_3}(z) = (2+2z^2) + 2z^2(2) = 2+6z^2$. The two-term recurrence relation above can be solved to get the closed form

$$\mathrm{IP}_{P_n}(z) = \left(\frac{1}{2} + \frac{3}{2\sqrt{1+8z^2}}\right)\left(\frac{1+\sqrt{1+8z^2}}{2}\right)^n + \left(\frac{1}{2} - \frac{3}{2\sqrt{1+8z^2}}\right)\left(\frac{1-\sqrt{1+8z^2}}{2}\right)^n$$

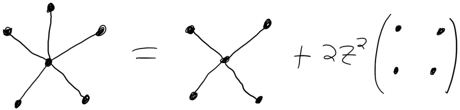
Example (n-Stars)



Let S_n be the star with $n+1$ positive vertices (i.e., the complete bipartite graph $K_{1,n}$). Then $\mathrm{IP}_{S_1}(z) = 2 + 2z^2$ and

$$\mathrm{IP}_{S_n}(z) = \mathrm{IP}_{S_{n-1}}(z) + 2z^2(2^{n-1}) = \mathrm{IP}_{S_{n-1}}(z) + 2^n z^2,$$

so we get the closed form $\mathrm{IP}_{S_n}(z) = (2^{n+1} - 2)z^2 + 2$. In fact, S_n are the only connected signed intersection graphs whose intersection polynomial is of the form $az^2 + b$ for positive $a, b \in \mathbb{Z}$.



Existence of Planar Partial Duals

Theorem 9

(Yan and Jin 2022): A bouquet B with more than one edge has a planar partial dual if and only if B is orientable and $\mathrm{I}(B)$ is bipartite.

Proof.

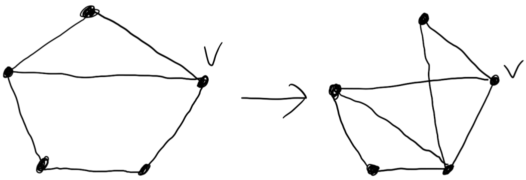
(of one direction): Suppose B is orientable and $I(B)$ is bipartite. Then there is a partition $V(I(B)) = X \sqcup Y$ such that every edge of $I(B)$ has one end in X and the other in Y. The corresponding partition $E(B) = X \sqcup Y$ is such that no ribbon edge of B intersects with a ribbon edge from the same cell. Then $\varepsilon(X) = \varepsilon(Y) = 0$, so $\varepsilon(B^X) = \varepsilon(X) + \varepsilon(Y) = 0$, i.e. B^X is planar. $\square$

Local Complementation

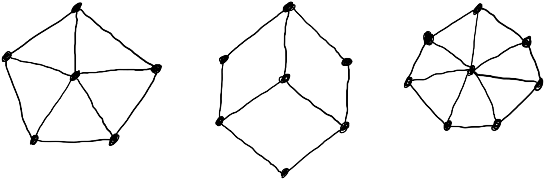
There is a concise classification of simple graphs that are not the intersection graph of any chord diagram. To state it, we need the following definition:

Definition 10

For a vertex v of a graph G, the **local complementation** of G at v is obtained by replacing the induced subgraph $G[N(v)]$ by its complementary graph.



Classification of Intersection Graphs

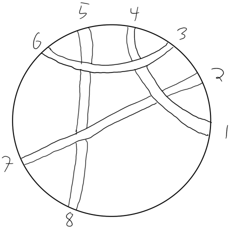


Theorem 11

(Bouchet 1994): A simple graph G is an intersection graph if and only if no graph obtained from G by successive local complementations has an induced subgraph isomorphic to one of the above graphs.

Specifying Ribbon Graphs by Permutations

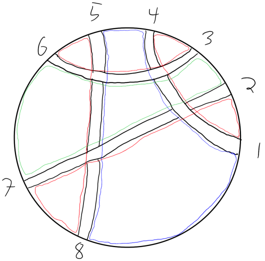
For computer computations, it is convenient to record an orientable ribbon graph R by two permutations σ and ρ on its set of half-edges, σ being the product of cycles giving the cyclic order of half-edges at each vertex, and ρ being the product of 2-cycles pairing the half-edges belonging to the same edge.



For the above bouquet, we have $\sigma = (1, 2, 3, 4, 5, 6, 7, 8)$ and $\rho = (1, 4)(2, 7)(3, 6)(5, 8)$.

Faces

The cycles of $\sigma\rho$ trace out the boundary components, so the number of faces of R is the number of cycles in $\sigma\rho$.



Here, we have $\sigma = (1, 2, 3, 4, 5, 6, 7, 8)$, $\rho = (1, 4)(2, 7)(3, 6)(5, 8)$, and $\sigma\rho = (1, 5)(2, 8, 6, 4)(3, 7)$.
I wrote some code in SageMath to calculate the partial-dual Euler genus polynomial of orientable bouquets here.

References

- André Bouchet, *Circle Graph Obstructions* (1994)
- Gross, Mansour, and Tucker, *Partial duality for ribbon graphs, I: Distributions* (2020)
- Yan and Jin, *Partial-dual polynomials and signed intersection graphs* (2022)