Recursion for the Partial-Dual Euler Genus Polynomial

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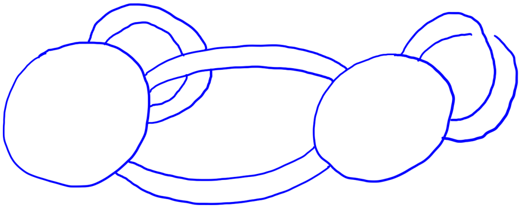
Ribbon Graphs

Definition

A *ribbon graph* $G = (V, E)$ is a surface with boundary expressed as the union of vertex and edge discs such that

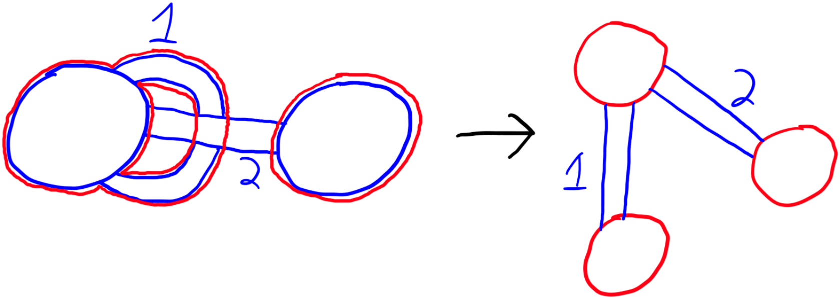
- Vertices and edges intersect at disjoint line segments
- Each such line segment lies on the boundary of exactly one vertex and edge
- Each edge contains exactly two line segments



Partial Duality

Definition (Chmutov, 2009)

For a ribbon graph G and $A \subseteq E(G)$, the *partial dual* G^A of G with respect to A is obtained by sewing a disc into each boundary component of the spanning ribbon subgraph induced by A and deleting the interiors of all vertices of G.



Partial duality at edge 1

Euler Genus

Definition

The *Euler genus* $\varepsilon(G)$ of a connected ribbon graph G is

$$\varepsilon(G) = \begin{cases} 2g(G) & \text{if G is orientable} \\ g(G) & \text{if G is non-orientable} \end{cases}$$

If G is not connected, then $\varepsilon(G)$ is the sum of the Euler genus of its components.

In terms of the Euler characteristic,

$$\chi(G) = v(G) - e(G) + f(G) = 2k(G) - \varepsilon(G).$$

Partial-Dual Polynomial

Definition (Gross, Mansour, and Tucker, 2020)

The *partial-dual Euler genus polynomial* of a ribbon graph G is

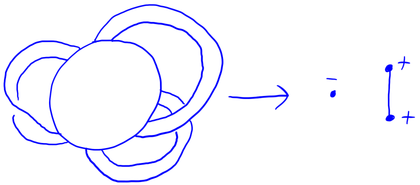
$$^{\partial}\varepsilon_G(z) = \sum_{A \subseteq E(G)} z^{\varepsilon(G^A)},$$

the generating function enumerating the edge subsets of G by the Euler genus of their partial duals.

Signed Intersection Graph

Definition

Given a bouquet (one-vertex ribbon graph) B, its *signed intersection graph* G is the graph whose vertices are the edges of B, such that two vertices are adjacent if and only if the corresponding ribbon edges intersect. The sign of a vertex is positive (negative) if its corresponding ribbon edge is untwisted (twisted).



If v is a vertex of G then $G^{\overline{v}}$ denotes the graph G with the opposite sign at v.

Intersection Polynomial

Theorem (Yan and Jin, 2022)

If two bouquets B_1 and B_2 have the same signed intersection graph, then $^{\partial}\varepsilon_{B_1}(z) = {}^{\partial}\varepsilon_{B_2}(z)$.

Thus we can define the *intersection polynomial* $\mathrm{IP}_G(z)$ of a signed intersection graph G as the partial-dual Euler genus polynomial of any bouquet whose signed intersection graph is G.

Leaf Recursion

Theorem (Yan and Jin, 2022)

Let G be a signed intersection graph and let v_1, v_2 be adjacent vertices of G such that v_1 is positive and has degree 1. Then

$$\mathrm{IP}_G(z) = \mathrm{IP}_{G-v_1}(z) + (2z^2)\,\mathrm{IP}_{G-v_1-v_2}(z).$$

This allows us to recursively compute the intersection polynomial of any positive tree.

Cut-vertex Recursion

Theorem (L.)

Let G be a signed intersection graph such that $G = G_1 \cup G_2$, $G_1 \cap G_2 = \{v\}$, and v is a cut-vertex of G. Let

$$\widetilde{Q_1} = \frac{2z^2 \, \mathrm{IP}_{G_2 - v} - (z+1)\, \mathrm{IP}_{G_2} + z\, \mathrm{IP}_{G_2^{\bar{v}}}}{2(2z^2 - z - 1)}$$

$$\widetilde{Q_2} = \frac{2z^2 \, \mathrm{IP}_{G_2 - v} - (z+1)\, \mathrm{IP}_{G_2^{\bar{v}}} + z\, \mathrm{IP}_{G_2}}{2(2z^2 - z - 1)}$$

$$\widetilde{Q_3} = \frac{z^2(\mathrm{IP}_{G_2} + \mathrm{IP}_{G_2^{\bar{v}}}) - 2(z^3 + z^2)\, \mathrm{IP}_{G_2 - v}}{2z^2 - z - 1}$$

If v is positive, then $\mathrm{IP}_G = \widetilde{Q_1}\, \mathrm{IP}_{G_1} + \widetilde{Q_2}\, \mathrm{IP}_{G_1^{\bar{v}}} + \widetilde{Q_3}\, \mathrm{IP}_{G_1 - v}$.

If v is negative, then $\mathrm{IP}_G = \widetilde{Q_2}\, \mathrm{IP}_{G_1} + \widetilde{Q_1}\, \mathrm{IP}_{G_1^{\bar{v}}} + \widetilde{Q_3}\, \mathrm{IP}_{G_1 - v}$.

A Corollary

Corollary (L.)

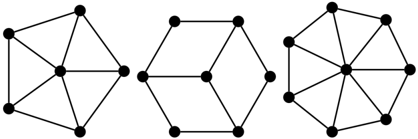
Let G be a signed intersection graph and v be a vertex of G. Then

$$\mathrm{IP}_{G-v}\left(-\frac{1}{2}\right) - \mathrm{IP}_{G}\left(-\frac{1}{2}\right) - \mathrm{IP}_{G^{\overline{v}}}\left(-\frac{1}{2}\right) = 0.$$

Moreover, if all vertices of G are positive, then

$$2z^2\,\mathrm{IP}_{G-v} + z\,\mathrm{IP}_{G} - (z+1)\,\mathrm{IP}_{G^{\overline{v}}} = 0.$$

Non-intersection Graphs



Theorem (Bouchet, 1994)

A simple graph is an intersection graph if and only if it does not have a vertex minor isomorphic to one of the above graphs.

Q: Can the intersection polynomial be extended to all graphs?

A: Yes, using the theory of delta-matroids.

Delta-Matroids

Definition

A *set system* is a pair $D = (E, \mathcal{F})$ of a set E and a collection $\mathcal{F}$ of subsets of E. We call D a *delta-matroid* if $\mathcal{F}$ is non-empty and satisfies the Symmetric Exchange Axiom:

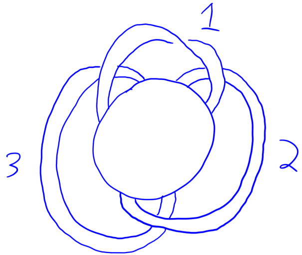
For all $X, Y \in \mathcal{F}$ and $u \in X \triangle Y$, there exists $v \in X \triangle Y$ such that $X \triangle \{u, v\} \in \mathcal{F}$.

E is called the ground set and elements of $\mathcal{F}$ are called feasible sets.

Ribbon-Graphic Delta-Matroids

Definition

From a ribbon graph G, we can obtain a delta-matroid $D(G) = (E, \mathcal{F})$ by taking E to be the edges of G and $\mathcal{F}$ to be the edge subsets corresponding to spanning ribbon subgraphs with exactly one boundary component. We call $D(G)$ a ribbon-graphic delta-matroid.



If G is the ribbon graph above, then

$$D(G) = (\{1, 2, 3\}, \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}).$$

Twist

Definition

Let $D = (E, \mathcal{F})$ be a delta-matroid and let $A \subseteq E$. Then the **twist** of D by A is the delta-matroid

$$D * A = (E, \{X \triangle A : X \in \mathcal{F}\}).$$

Proposition

Let G be a ribbon graph and let $A \subseteq E(G)$. Then

$$D(G^A) = D(G) * A.$$

Width

Definition

Let $D = (E, \mathcal{F})$ be a delta-matroid. The **width** of D is

$$w(D) = \max\{|F| : F \in \mathcal{F}\} - \min\{|F| : F \in \mathcal{F}\}.$$

Proposition

Let G be a ribbon graph. Then

$$\varepsilon(G) = w(D(G)).$$

Twist Polynomial

Definition (Yan and Jin, 2021)

Let $D = (E, \mathcal{F})$ be a delta-matroid. Then the *twist polynomial* of D is

$$^{\partial}w_D(z) = \sum_{A \subseteq E} z^{w(D*A)}.$$

From the previous two propositions, it is immediate that if G is a ribbon graph, then $^{\partial}\varepsilon_G(z) = {}^{\partial}w_{D(G)}$.

Binary Delta-Matroids

Definition

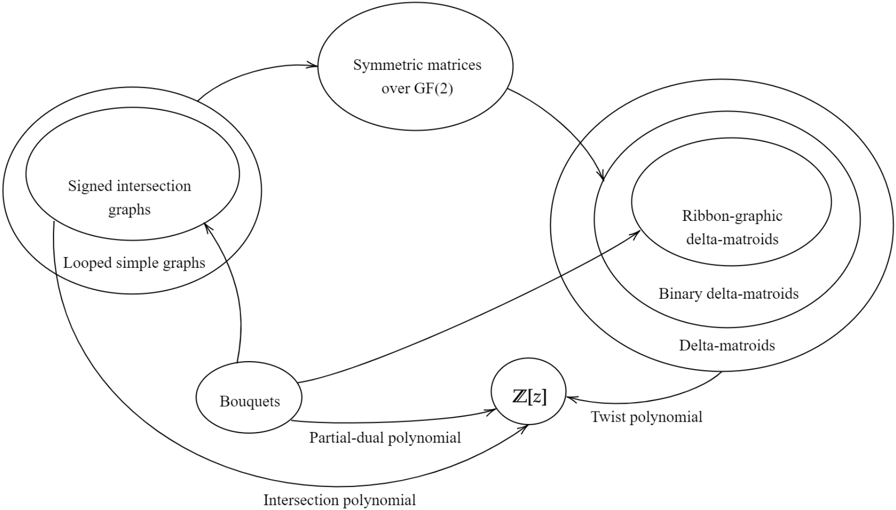
Let E be a finite set and M be a symmetric matrix over GF(2) with rows and columns indexed by E. Let $D(M) = (E, \mathcal{F})$, where

$$\mathcal{F} = \{A \subseteq E : M[A] \text{ is invertible}\}.$$

We take $M[\emptyset]$ to be invertible. Then $D(M)$ is a delta-matroid and $D * A$ for any $A \subseteq E$ is called a *binary delta-matroid*.

M can be viewed as the adjacency matrix of a looped simple graph.

A Commutative Diagram



Adjacency Polynomial

Definition

Let G be a looped simple graph. We define the *adjacency polynomial* $\mathrm{AP}_G(z)$ to be the twist polynomial of the binary delta-matroid obtained from the adjacency matrix of G.

Theorem (L.)

The cut-vertex recurrence relation for the intersection polynomial extends to the adjacency polynomial.