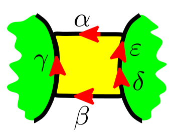
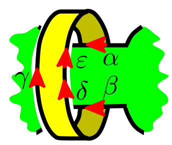
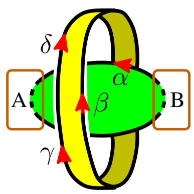
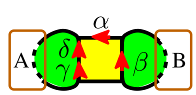
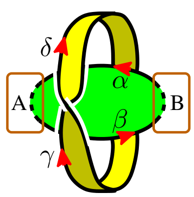
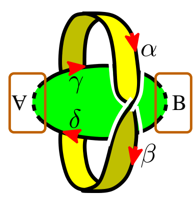
Partial duality of ribbon graphs [Ch]

Definition. For a ribbon graph G and a subset of the edge-ribbons $D \subseteq E(G)$, the ***partial dual***, G^D of G relative to D is a ribbon graph constructed as follows.

- The vertex-discs of G^D are bounded by connected components of the boundary of the spanning subgraph of G containing all the vertices of G and only the edges from D.
- The edge-ribbons of $E(G) \setminus D$ are attached to these new vertices exactly at the same arcs as in G. The edge-ribbons from D become parts of the new vertex-discs. For $e \in D$ we take a copy of e, e', and attach it to the new vertex-discs in the following way. The rectangle representing e intersects with vertex-discs of G by a pair of opposite sides. But it intersects the boundary of the spanning subgraph, that is the new vertex-discs, along the arcs of the other pair of its opposite sides. We attach e' to these arcs. The copies of the first pair of sides in e' become arcs of the boundary of G^D.

e	G	G^e
non loop		
orientable loop		
non-orientable loop		

Properties of partial duality.
Let G be a ribbon graph.

(a) Suppose $E(G) \ni e \notin D \subseteq E(G)$. Then $G^{D\cup\{e\}} = \left(G^D\right)^{\{e\}}$.

(b) $\left(G^D\right)^D = G$.

(c) $\left(G_1^D\right)^{D_2} = G^{\Delta(D_1,D_2)}$, where $\Delta(D_1,D_2) := (D_1 \cup D_2) \setminus (D_1 \cap D_2)$ is the symmetric difference of sets.

(d) Partial duality preserves orientability of ribbon graphs.

(e) Let $\widetilde{G}$ be a surface without boundary obtained from G by gluing discs to all boundary component of G. Then $\widetilde{G^D} = \widetilde{G^{E(G)\setminus D}}$.

(f) The partial duality preserves the number of connected components of ribbon graphs.

Partial-dual Genus Polynomial [GMT]. $\displaystyle {}^{\partial}\Gamma_G(z) := \sum_{D\subseteq E(G)} z^{g(\widetilde{G^D})}$.

Arrow dichromatic polynomial [BBC].

$$A_G(a,\mathbf{b},c,\mathbf{K}) := \sum_{F\subseteq E(G)} a^{k(F)}\Big(\prod_{e\in F} b_e\Big)c^{bc(F)} \prod_{f\in\partial(F)} K_{i(f)}$$

$$Z_G(q,v) = A_G(a,\mathbf{b},c,\mathbf{K})|_{a=q,b_e=v,c=1,K_i=1}$$

Arrow Bollobás-Riordan polynomial [BBC].

$$BR_G(X,Y,Z,\mathbf{K}) := \sum_{F\subseteq E(G)} (\prod_{e\in F} x_e)\ (\prod_{e\notin F} y_e)\ X^{r(G)-r(F)}\ Y^{n(F)}\ Z^{k(F)-bc(F)+n(F)} \prod_{f\in\partial(F)} K_{i(f)}.$$

$$BR_G(X,Y,Z,\mathbf{K}) = \Big(\prod_{e\in E(G)} y_e\Big)(YZ)^{-v(G)}X^{-k(G)}A_G(XYZ^2,\{x_eYZ/y_e\},Z^{-1},\mathbf{K})\ .$$

The partial duality property at $a = 1$. For $D \subseteq E(G)$

$$A_G(1,\mathbf{b},c,\mathbf{K}) = \Big(\prod_{e\in D} b_e\Big)A_{G^D}(1,\mathbf{b}_D,c,\mathbf{K})\ , \qquad \text{where } \mathbf{b}_D = \{b_e^D\}, b_e^D = \begin{cases} b_e & \text{if } e\notin D\ , \\ 1/b_e & \text{if } e\in D\ . \end{cases}$$

Arrow Thistlethwaite theorem [BBC]**.** Let L be a virtual link diagram and let G_L^s be the signed arrow ribbon graph corresponding to a state s with e_- negative edges and e_+ positive edges. Then the arrow bracket polynomial of L is a specialization of the arrow dichromatic polynomial of G_L^s:

$$[L]_A(A, B, d, \mathbf{K}) = \frac{A^{e_+}B^{e_-}}{d} A_{G_L^s}(1, \mathbf{b}, d, \mathbf{K}) \ ,$$

where the weight variables are specialized to $b_e = \begin{cases} B/A & \text{if } e \text{ is positive,} \\ A/B & \text{if } e \text{ is negative.} \end{cases}$

References

[BR] B. Bollobás and O. Riordan, *A polynomial of graphs on surfaces*, Math. Ann. **323** (2002) 81–96.

[BBC] R. Bradford, C. Butler, S. Chmutov, *Arrow ribbon graphs*, Journal of Knot Theory and its Ramifications, **21**(13) (2012) # 1240002 (16 pages).

[Ch] S. Chmutov, *Generalized duality for graphs on surfaces and the signed Bollobás-Riordan polynomial*, Journal of Combinatorial Theory, Ser. B **99**(3) (2009) 617–638; preprint `arXiv:math.CO/0711.3490`.

[GMT] J. L. Gross, T. Mansour, T. W. Tucker, *Partial duality for ribbon graphs, I: Distributions*, European Journal of Combinatorics **86**, 1–20 (2020) 103084