# Partial-Dual Genus Polynomial

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# Ribbon Graph

## Definition 1

A **ribbon graph** G is a surface (possibly non-orientable) with boundary, represented as the union of two sets of closed topological discs called **vertices** V (G) and **edges** E(G), satisfying the following conditions:

- these vertices and edges intersect by disjoint line segments;
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;

One can think a ribbon graph as a set of discs(vertices) and a set of rectangles(edges), with each rectangle(edge) glued to two (possibly the same) discs along two opposite sides.

# Partial Duality

**Definition 2**

Let $G$ be a ribbon graph and $A \subseteq E(G)$. Glue a disc to each boundary component of the spanning ribbon subgraph $(V(G), A)$ and remove the interior of all vertices of $G$. The resulting ribbon graph is called the *dual graph with respect to the subset* $A$, and is denoted by $G^A$.

# Partial-Dual Genus Polynomial

## Definition 3

The **partial-dual genus polynomial** of any ribbon graph G is the generating function

$$^{\partial}\varepsilon_G(z) = \sum_{A \subseteq E(G)} z^{\varepsilon(G^A)}$$

that enumerates partial duals by Euler-genus.

Where $\varepsilon(G^A) := \varepsilon(\widetilde{G^A})$, and $\widetilde{G^A}$ is the surface obtained by gluing a disc to each boundary component of $G^A$.

A **bouquet** is a ribbon graph with only one vertex.

For a connected ribbon graph $G$, oen can take $B$ to be a set of edges such that $B \sqcup V(G)$ is a tree. Since $B \sqcup V(G)$ is a tree, $G^B$ has only one vertex. Thus $G^B$ is a bouquet. In addition, since $(G^{E_1})^{E_2} = G^{E_1 \Delta E_2}$ for any $E_1$, $E_2 \subseteq E(G)$ (Property(b) of the 5th handout), we have

$$^{\partial}\varepsilon_{G^B}(z) = \sum_{A \subseteq E(G)} z^{\varepsilon((G^B)^A)} = \sum_{A \subseteq E(G)} z^{\varepsilon(G^{B\Delta A})} = \sum_{A' \subseteq E(G)} z^{\varepsilon(G^{A'})} = {}^{\partial}\varepsilon_G(z).$$

The second last equality comes from the fact that the power set of any set is an abelian group under symmetric difference. In this sense, it suffices to study partial-dual genus polynomials for bouquets.

# Propositions that Help Computations

Let $v(G)$, $e(G)$, $f(G)$ be the number of vertices, the number of edges, and the number of boundary components, respectively.

## Proposition 1

Let $B$ be a bouquet. Then the Euler genus $\varepsilon(B)$ is given by the equation

$$\varepsilon(B) = 1 + e(G) - f(G)$$

*Proof.* $2 - \varepsilon(B) = v(G) - e(G) + f(G)$. For a bouquet, $v(G) = 1$

## Proposition 2 [GMT20]2.3

Let $B$ be a bouquet, and let $A \subseteq E(B)$. Then

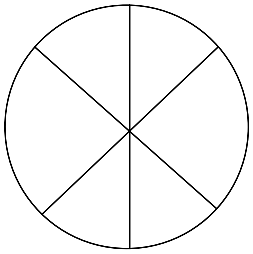
$$\varepsilon(B^A) = \varepsilon(A) + \varepsilon(A^c)$$

where $A^c = E(B) - A$ and $\varepsilon(A)$ is the Euler genus of the subgraph induced by A.

[GMT20] J. L. Gross, T. Mansour, T. W. Tucker, *Partial duality for ribbon graphs, I: Distributions*, European Journal of Combinatorics 86 (2020) 103084, 1–20.

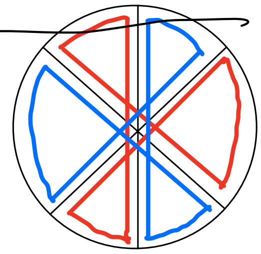
$$\mathcal{E}(G^A) = \mathcal{E}(A) + \mathcal{E}(A^c) = \mathcal{E}(G^{A^c})$$

$\mathcal{E}(A) = 1 + e(A) - f(A)$

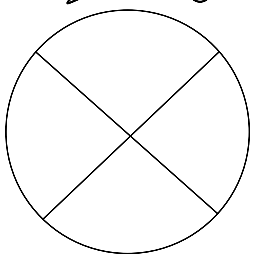
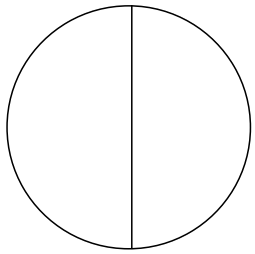


$z^2$

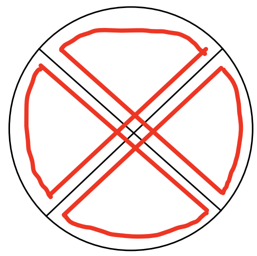
$2 + 0 = 2$



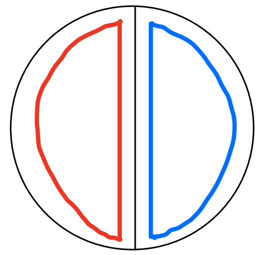
$1 + 3 - 2 = 2$

$3 \times$  

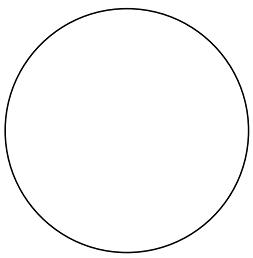
$2 \quad 3z^2 \qquad 3z^2$



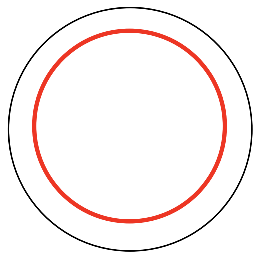
$1 + 2 - 1 = 2$



$1 + 1 - 2 = 0$



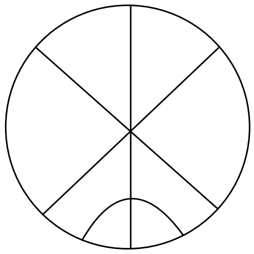
$z^2$



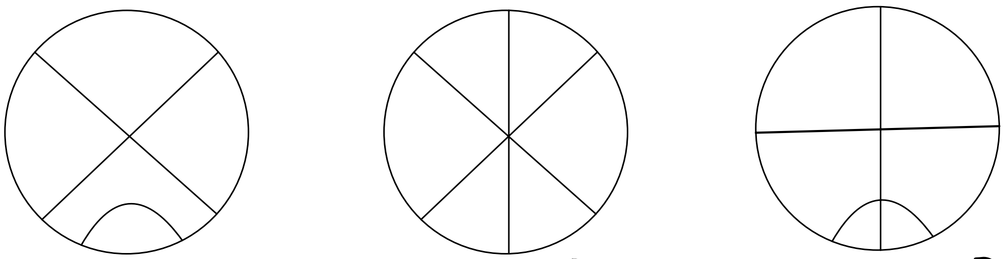
$1 + 0 - 1 = 0$

$$\partial_{\circledast}(z) = 8z^2$$

$$\partial_{\otimes}(z) = 2(z^4 + 4z^2 + z^4 + 2z^2) = 4z^4 + 12z^2$$

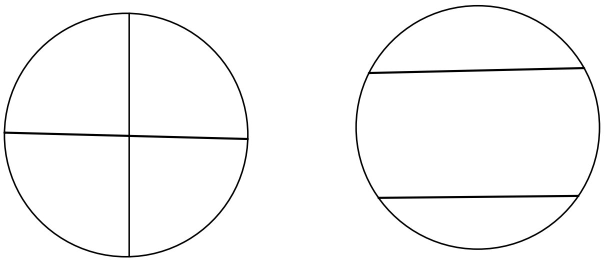


$z^4$

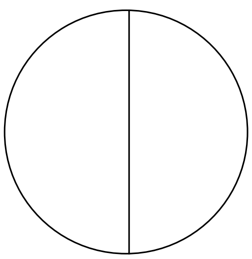


$z^2$ $z^2$ $2z^2$ $\times 2$

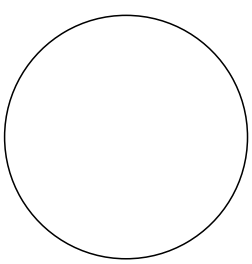
$\times 4$

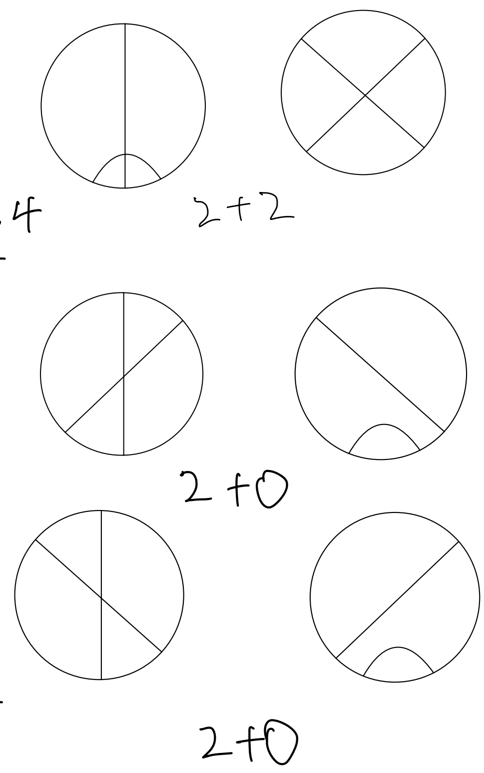


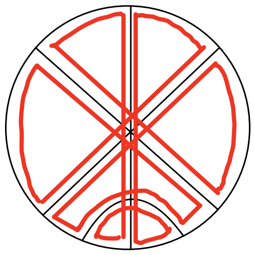
$\times 2$



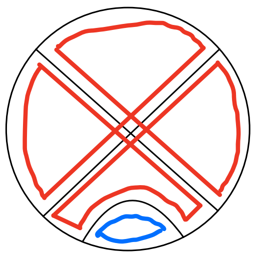
$\times 6$

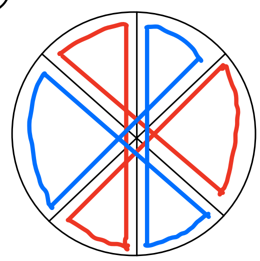


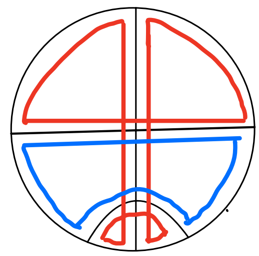




$1+4-1=4$



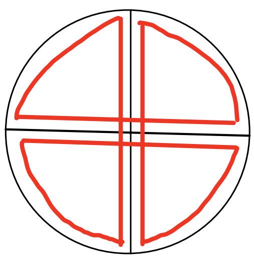


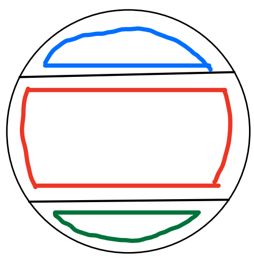


×2

$1+3-2=2$ $1+3-2=2$ $1+3-2=2$

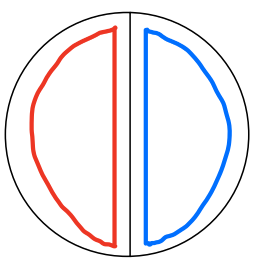
×4





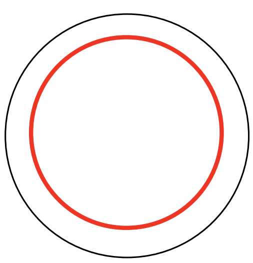
× 2

$1+2-1=2$ $1+2-3=0$



× 6

$1+1-2=0$



$1+0-1=0$