Knots and Graphs

Working Group [Summer 2024]

MATH 4193, class number 16239
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RESEARCH PROJECTS

Project 1. ***Polynomial invariants of links in thickened surfaces.*** (Gabriel Black, David Kruzel, Adarsh Praturi. TA: Luke Wiljanen.)

In [Bon, Theorem 6.2] Joe Boninger introduced two Jones-type polynomials for a checkerboard-colorable link in a thickened surface using the Tait graphs of its checkerboard diagram and their Tutte-type polynomials. Such a link may be regarded as virtual link of of L. Kauffnam [Ka]. For virtual links one the most general polynomial invariant is the *arrow polynomial* [DK, BBC]. The goal of the project is to try to get Boninger's polynomials as specialization of the arrow polynomial or show that these polynomials are independent.

A general standard excellent introduction to knot theory is [Ad].

Project 2. ***Partial-dual genus polynomial.*** (Wo Wu, Charlton Li. TA: Jake Huryn)

Partial duality of ribbon graphs relative to a subset of its edges was introduced in [Ch09]. This operation often changes the genus of the ribbon graph. The classical Euler–Poincaré duality is the partial duality relative to all the edges. For an excellent exposition see [EMM13]. J. L. Gross, T. Mansour, and T. W. Tucker [GMT20] introduced the partial-dual genus polynomial as a genus generating function for all partial duals of a given ribbon graph.

Chord diagrams are the main combinatorial object of the theory of Vassiliev knot invariants (see, for example [CDM12]). We can consider a chord diagram as an oriented ribbon graph with a singe vertex given by the circle of the chord diagram and ribbon-edges corresponding to its chords. The four-term relation for functions on chord diagrams is the key relation in the theory of Vassiliev knot invariants. The functions on chord diagram satisfying it are called *weight systems*.

Recently it was shown [Ch23] that the partial-dual genus polynomial of one-vertex ribbon graphs is a weight system on the corresponding chord diagrams. Moreover in [YaJi22s] it was shown that it depends only on the *intersection graph* of a chord diagram. The proof is based on an explicit recurrence formula in the Theorem 5.2. This formula looks similar to the formula in the Property 1 of [BM] appearing in a completely different aspect of knot theory.

We will try to find any relation between these formulas and consequently extend the recurrent formula for this polynomial from the intersection graphs of chord diagrams to all graphs.

Project 3. ***Chromatic Polynomials of 2-Edge-Colored Graphs.*** (Jeremy Case, Steven Wu. TA: Jake Huryn)

Recently a chromatic polynomial of 2-edge colored graph was introduced in [BCDZ]. Such graph can be considered as a signed graphs, where a sign + or − is assigned to each edge. For signed graphs there are two chromatic polynomials defined in [Za]. But they are different from the polynomial of [BCDZ]. The goal of this project is to try try to find a relation between all of these polynomials, the Tutte-type generalization of [BCDZ], and look for possible applications of these polynomials in knot theory.

References

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