Arrow Polynomial of Mod k Almost Classical Virtual Links

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Summary of Previous Results

Recall the concept of an Alexander numbering, which requires every arc to be labeled with an integer so that locally each crossing looks like one of the following:

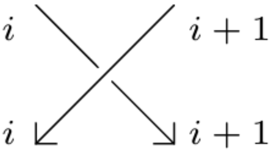

Figure: Alexander Numbering

Summary of Previous Results

We will be focusing on mod k Alexander numberings, which take labels from $\mathbb{Z}/k\mathbb{Z}$ instead of the normal integers. For example, the labels in the figure follow the rules for a mod 5 Alexander numbering around a crossing, since $4 + 1 = 0$ in this setting.

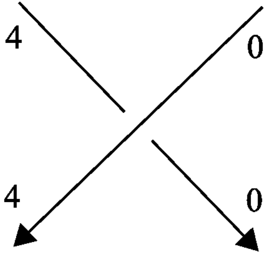

Figure: Valid Mod 5 Labels

Summary of Previous Results

Previously, we stated the following result on restrictions of the arrow polynomial of a mod k almost classical virtual link:

Theorem

Let D be a mod k almost classical virtual link. Then, all "extra variables" in the arrow polynomial of D must have an index which is a multiple of k if k is odd or a multiple of $\frac{k}{2}$ if k is even.

Summary of Previous Results

The main idea behind the proof was to realize that every fully simplified loop in an expanded state S of virtual link D with a mod k Alexander numbering, the loop is made up of repeated pairs of poles as in the figure below (possibly rotated).

Figure: Possible Pairs of Poles not Canceling

For the pairs to survive, either the left type or the right type will repeat several times.

Now, notice that going from left to right if we add i to the number of poles we have passed so far and use that as our label, it should agree with the actual label after reduction mod k. So, if $2n$ is the number of poles on the loop, it must satisfy $i + 2n \equiv i \mod k$ since the two ends of the loop must meet up.

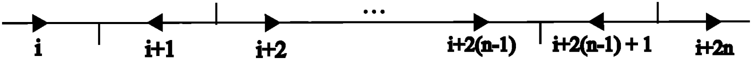

Figure: Fully Simplified Loop

Summary of Previous Results

Now, recall that the index of an extra variable corresponding to a loop is half the number of poles on that loop after cancellation. So, if $2n$ is the number of poles on the loop, n is the index of the extra variable. It satisfies $i + 2n \equiv i \mod k$ or equivalently $2n \equiv 0 \mod k$.

If k is odd, then 2 is invertible in the ring $\mathbb{Z}/k\mathbb{Z}$, so we can divide by 2 on both sides to get $n \equiv 0 \mod k$, meaning n must be a multiple of k.

If k is even, then this is effectively the same as requiring $n \equiv 0 \mod \frac{k}{2}$, meaning n must be a multiple of $\frac{k}{2}$. Since all extra variables originate from loops, this completes the proof.

Greater Restrictions on the k = 2 Case

We first will introduce a few definitions to make future statements easier.

Definition

Given a virtual link D, note that each summand of the arrow polynomial is of the form $A^s(K_{i_1}^{j_1} K_{i_2}^{j_2} \ldots K_{i_v}^{j_v})$. For such a summand, we define the K-degree to be $i_1 j_1 + i_2 j_2 + \ldots + i_v j_v$. Also, we let $AS(D)$ denote the set of K-degrees of summands of the arrow polynomial of D.

Greater Restrictions on the k = 2 Case

In a 2021 paper, Deng, Jin, and Kauffman proved the following theorem (restated slightly):

Theorem

Let D be a mod 2 almost classical virtual link. Then,

(1) $AS(D)$ *contains only even integers and*

(2) For any summand $A^s K_{i_1}^{j_1} K_{i_2}^{j_2} \dots K_{i_v}^{j_v}$ *of the arrow polynomial of D where the sequence* $(i_n)_{n=1}^{v}$ *is strictly increasing and each* j_n *is a positive integer, we have* $2i_v \leq \sum_{t=1}^{v} i_t j_t$. *In other words, the K-degree of any summand is at least twice the largest index of an extra variable.*

A Generalization

We will state and prove a generalization of this theorem:

Theorem

Assume D is a mod $2k$ almost classical virtual link for some positive integer k and S is a fully expanded state of D. Let T be the multiset containing each of the loop indices of S. Then, there is some partition of T into two multisets A and B such that $\sum_{a \in A} a = \sum_{b \in B} b$.

A Generalization

We will first show how this theorem implies the theorem of Deng, Jin, and Kauffman.

The most important fact to notice is that the K-degree of a summand is exactly the sum of all loop indices of the state from which the summand originated. This means that (1) is equivalent to saying that $\sum_{t \in T} t$ is even in the notation from the generalization.

If the generalization is true, then we must have $\sum_{t \in T} t = \sum_{a \in A} a + \sum_{b \in B} b = 2 \sum_{a \in A} a$, an even number.

A Generalization

Statement (2) from the first theorem also follows from the generalization. We will consider the contrapositive. Assume that $2i_v > \sum_{t=1}^{v} i_t j_t$. This is the same as saying that $2 \max_{t \in T}(t) > \sum_{t \in T} t$. Equivalently, $\max_{t \in T}(t) > \sum_{a \in A} a = \sum_{b \in B} b$. However, this contradicts the statement of the generalization, since in this case it is impossible for the maximum element of T to belong to A or B.

Proof of Generalization

We will examine the two cases $2k = 2$ and $2k = 4$ (recall that $2k$ is the modulus of the Alexander numbering) to demonstrate the general argument.

For the case $2k = 2$, we notice that there are two distinct types of crossings in the virtual link. One type has $i = 0$ and the other has $i = 1$ when labeling the crossing as below.

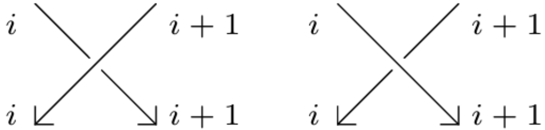

Figure: Alexander Numbering

Proof of Generalization

Every time poles are created, they are created in pairs. In our mod 2 setting, the two poles created must look like the two on the left or like the two on the right. Since these are always created together, before cancellation there must be an equal number of poles which are vertically adjacent in the figure.

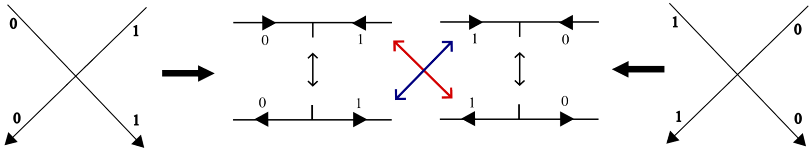

Figure: Two Expansion Types

Proof of Generalization

Next, we examine how pairs of poles can be arranged next to each other. Each pole can only be next to a pole connected to it by an arrow in the figure. Poles connected by black arrows always cancel when they are adjacent and poles connected by colored arrows never cancel each other. In particular, this means that even as poles cancel, there will continue to be the same number of poles which are vertically adjacent in the figure.

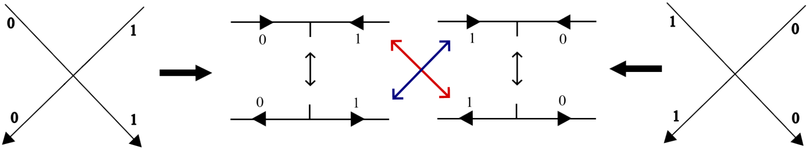

Figure: Two Expansion Types

Proof of Generalization

Now, considering a fully simplified loop in a state S of D, we note that only poles of one color may remain. If two poles of different colors were adjacent, they would have to be vertically adjacent poles, which always cancel each other. Also, recalling that poles always come in pairs, we can note that each pole of a given color shows up the same number of times on each simplified loop.

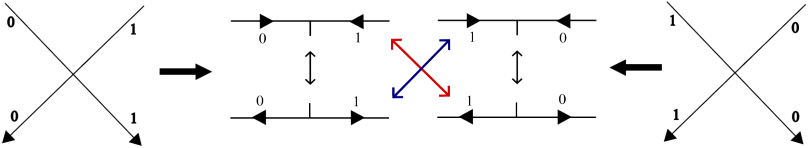

Figure: Two Expansion Types

Proof of Generalization

Now, since each loop only contains one color of pole, the coloring of poles induces a coloring on the loops of S. We use this coloring to split the loop indices of S into multisets A and B. Any loop with red poles has its index assigned to A and any loop with blue poles has its index assigned to B.

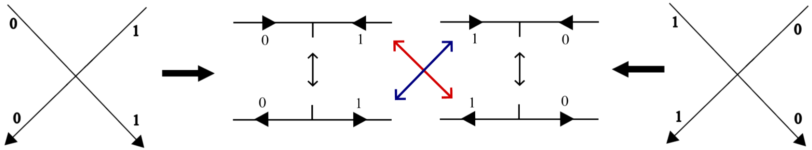

Figure: Two Expansion Types

Proof of Generalization

Finally, recalling that vertically adjacent pole types have equal number after cancellation, we get that there are an equal number of blue poles as red poles. This will give us our desired result that $\sum_{a \in A} a = \sum_{b \in B} b$.

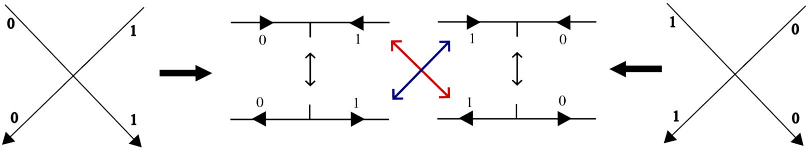

Figure: Two Expansion Types

Proof of Generalization

Next, we'll look at the $2k = 4$ case, which illustrates an argument that works for any even number. Much of the same analysis holds. Poles are still always created in pairs which are vertically adjacent in the figure, and canceling poles are always vertically adjacent, so the number of poles of vertically adjacent pairs still remains equal throughout the process.

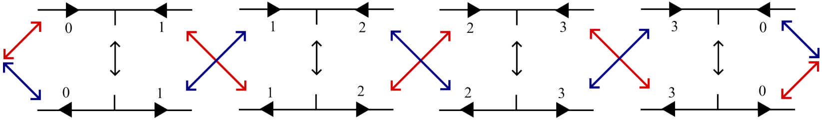

Figure: Four Expansion Types

Proof of Generalization

The key difference is that each pole can now be adjacent to two different types of poles without canceling. However, after cancellations, moving through the poles on a loop in order will correspond to moving one step rightward each time or one step leftward each time (wrapping around as necessary) following the colored arrows.

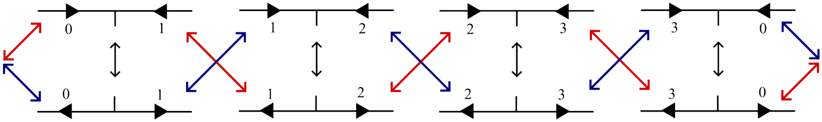

Figure: Four Expansion Types

Proof of Generalization

Thus, we still get an induced coloring on the loops of state S and can use that coloring to partition the loop indices into multisets A and B based on color. Since the number of poles is the same across vertical pairs, there are the same number of red and blue poles, and thus we again get our desired result.

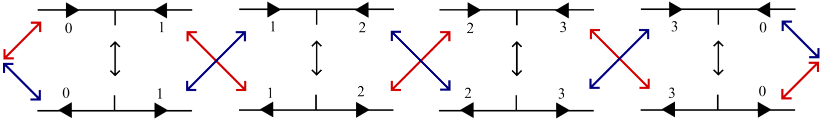

Figure: Four Expansion Types

Interesting Observations

While it wasn't necessary for the proof of the theorem, we can combine this analysis with the results from the summary section (loop indices are all multiples of the half the modulus) to observe more about the simplified states of D. Recall that the number of poles on a loop is twice the loop index, so the number of poles on a loop must be a multiple of the modulus.

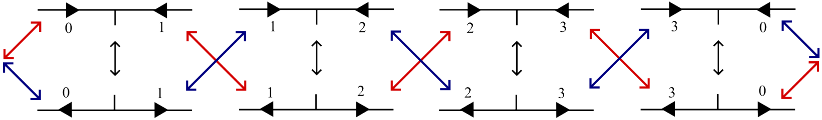

Figure: Four Expansion Types

Interesting Observations

In the context of a mod $2k$ almost classical virtual link, this means that every type of pole of the same color must show up an equal number of times on every loop. Combined with the earlier observation that vertically adjacent poles occur in equal number, we get that every distinct pole type must occur in equal number.

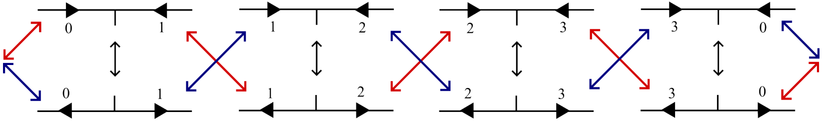

Figure: Four Expansion Types

Interesting Observations

In fact, this observation is not restricted to even moduli. For odd moduli, the number of poles must be a multiple of the modulus times two. This means that after cancellation the poles on any loop must cycle through every distinct pole type, containing an equal number of each of them.

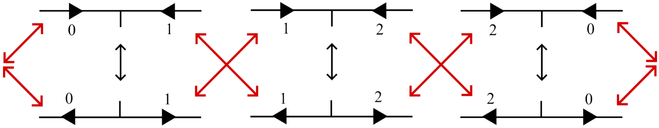

Figure: Three Expansion Types

Consequences and Examples

Any virtual link which is mod k almost classical is also mod d almost classical for any d which is a divisor of k. Using our result from the previous presentation, we can show that the converse is false. The following virtual link has arrow polynomial $-A^2K_1^2 - A^{-2}K_1^2$. As shown, it is mod 2 almost classical, but it cannot be mod 4 almost classical since 1 is not a multiple of $\frac{4}{2}$.

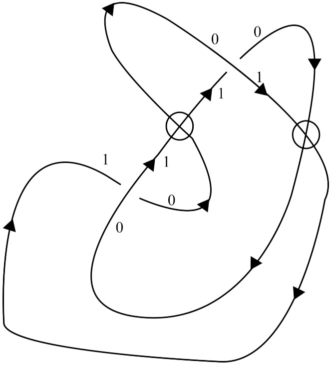

Figure: Mod 2 Almost Classical but Not Mod 4 Almost Classical

Consequences and Examples

This theorem allows us to rule out more virtual links from being mod $2k$ classical than the theorem from the previous presentation. For example, virtual knot 4.55 pictured to the right has arrow polynomial $A^4 + A^{-4} + 1 - (A^4 + 2 + A^{-4})K_1^2 + 2K_2$. The final summand corresponds to the multiset $\{2\}$, which cannot be partitioned as required. It also violates (2) of the theorem of Deng, Jin, and Kauffman.

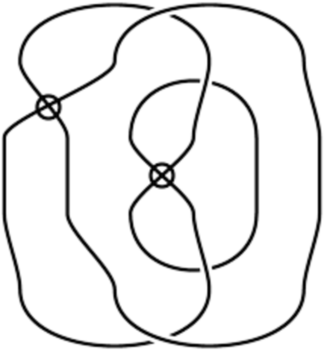

Figure: Virtual Knot 4.55

Consequences and Examples

The generalization of the theorem of Deng, Jin, and Kauffman has the potential to rule out more virtual links from being mod $2k$ almost classical than the original theorem can. However, such a virtual link would have to have at least six classical crossings since the smallest K-degree summand which is distinguished by the original theorem and the generalization is $A^s K_2^3$. Such an example has not yet been found.

References

- Deng, Qingying & Jin, Xian'an & Kauffman, Louis. (2021). On arrow polynomials of checkerboard colorable virtual links. Journal of Knot Theory and Its Ramifications. 30. 2150053. 10.1142/S021821652150053X.
- Green, Jeremy. "A Table of Virtual Knots." Virtual Knot Table, www.math.toronto.edu/drorbn/Students/GreenJ/. Accessed 6 July 2025.