The Penrose Bracket and it's Extension

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Outline

Tensors

Definition

A tensor T is a set of numbers $\{T_{abc...}\}$ indexed by $\{a, b, c, \dots\} \in \{1, \dots, n\}$.

Example (Levi-Civita symbol)

The Levi-Civita symbol is the tensor defined by:

$$\varepsilon_{abc} := \begin{cases} 1, & \text{if } (a, b, c) \text{ is an even permutation of } (1, 2, 3) \\ -1, & \text{if } (a, b, c) \text{ is an odd permutation of } (1, 2, 3) \\ 0, & \text{if } a = b,\ b = c \text{ or } c = a \end{cases}$$

Tensors

Example (Cross Product)

The components of the cross product of

$$v := t \times u = \begin{bmatrix} t_2 u_3 - t_3 u_2 \\ t_3 u_1 - t_1 u_3 \\ t_1 u_2 - t_2 u_1 \end{bmatrix}$$

can be defined in terms of the Levi Civita symbol:

$$v_a = \varepsilon_{abc} t_b u_c$$

Tensors

Example (Kronecker delta)

The Kronecker delta is the tensor defined by:

$$\delta_{ab} := \begin{cases} 1, & a = b \\ 0, & a \neq b \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example (Identity Matrix)

The identity matrix in n dimensions I_n, is the tensor whose components are defined by the Kronecker delta:

$$(I_n)_{a,b} = \delta_{ab}$$

Tensors

Definition (Tensor Contraction)

The contraction of 2 tensors $T_{...b...}$ and $U_{...b...}$ by index b is the summation of their product over index b:

$$V_{...} = \sum_b T_{...b...} U_{...b...}$$

Example (Matrix Multiplication)

Matrix multiplication between $M = \{M_{ab}\}$ and $N = \{N_{bc}\}$, with $O := MN$ can be viewed as a contraction over b:

$$O_{ac} = \sum_b M_{ab} N_{bc}$$

Tensors

$\cancel{\hat{i}\cdot(\vec{b}\times\vec{i}) = (a\cdot b)}$

Lemma (Levi Civita Symbol Identity)

The Levi Civita symbol satisfies the equation

$\nabla \times (\nabla \times \vec{V})$

$$\sum_{c=1}^{3} \varepsilon_{abc}\varepsilon_{cde} = \delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}$$

Proof: Suppose $a = b$ and $d \neq e$, then the LHS is 0 because $\varepsilon_{abc} = 0$ and the RHS is 0 because d or e must be different than $a = b$. The same applies for the case $a \neq b$ and $d = e$. When both $a = b$ and $d = e$ the RHS will become $1 - 1 = 0$ when $a = d$ or $0 - 0 = 0$ otherwise.

Tensors

For the cases where $a \neq b$ and $d \neq e$, if $\{a, b, d, e\} = \{1, 2, 3\}$ then by the pidgeon hole principle ε_{abc} or ε_{cde} is 0, making the LHS 0. The RHS is also $0 - 0 = 0$ because if any of the terms were 1, then $\{a, b, d, e\}$ would only have 2 elements. The other cases is when there exists a unique c such that $\varepsilon_{abc}\varepsilon_{cde} \neq 0$. If $a = d$ then $b = e$ and ε_{abc} has the same sign as ε_{cde} so the LHS is 1, the same as the RHS. If $a \neq d$ so that $a = e$ and $b = d$,ε_{abc} has the opposite sign as ε_{cde} so the LHS is -1, the same as the RHS.

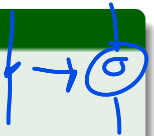
Tensor Diagrammatics

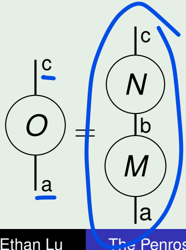
Definition

A tensor network is a graph in which the vertices represent the tensors and the edges are indices, with the shared indices contracted.

Example (Matrix Multiplication)

The formula $O_{ac} = \sum_b M_{ab} N_{bc}$ can be represented diagrammatically as:



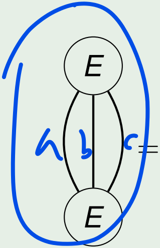


Tensor Diagrammatics

Definition (Epsilon Network)

A epsilon network is a tensor network with each tensor being $E_{abc} := \underline{i\varepsilon_{abc}}$ where $i = \sqrt{-1}$

Example



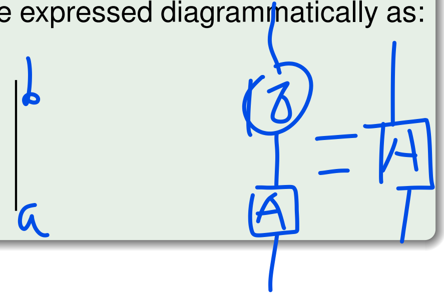
$= 6$

$$E_{abc}E_{cba} = -(E_{abc})^2 = (\varepsilon_{abc})^2$$

Tensor Diagrammatics

Example (Kronecker Delta)

The Kronecker Delta δ_{ab} can be expressed diagrammatically as:

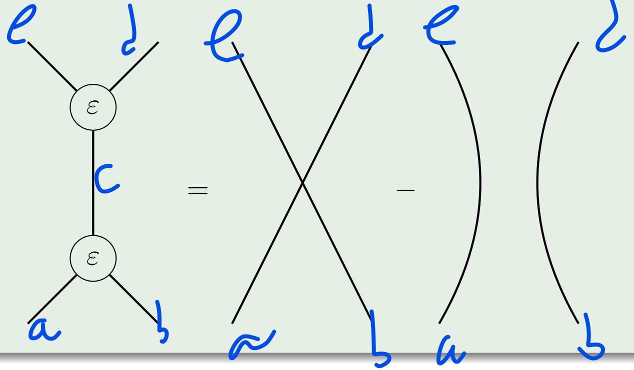


Tensor Diagrammatics

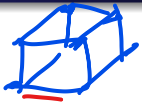
$$\sum \varepsilon_{abc}\varepsilon_{cde} = \delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}$$

Example (Levi Civita Symbol Identity)

The previous lemma can be expressed diagrammatically as:



proper colorings of planar cubic graphs



Definition

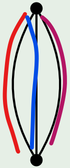
A planar cubic graph is a graph such that each vertex has 3 edges and has an immersed into the plane without any edge crossings

Definition

A proper 3-coloring is an assignment of 3 colors to the edges of a graph such that at every vertex there are no 2 repeated colors

proper colorings of planar cubic graphs

Example

 is a planar cubic graph with 6 proper 3-colorings.

Penrose Evaluation

$$\epsilon_{abc} = i\,\epsilon_{abc}$$

Theorem (Penrose)

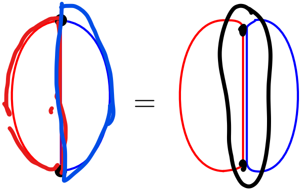
The number of proper 3-colorings of a planar cubic graph is equal to the contraction of it's corresponding Epsilon-network

Penrose Evaluation

Definition

A formation is a finite collection of simple closed curves in the plane colored either red or blue.

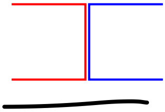
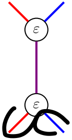
A formation can be associated to a proper 3-coloring of a cubic graph as follows:

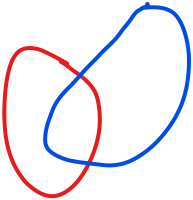


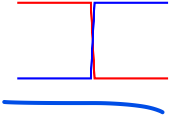
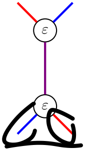
Penrose Evaluation

Given a coloring, the contraction can then be evaluated by the ways the red and blue formations intersect:

$\varepsilon_{rbp}\varepsilon_{pbr}$

 $= -$  $= 1$ $= -(\varepsilon_{rbp})^2$ $= -1$



 $= -$  $= -1$

Because formations are simple and closed, the number of crossings must be even.

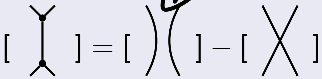
Penrose Bracket

Theorem

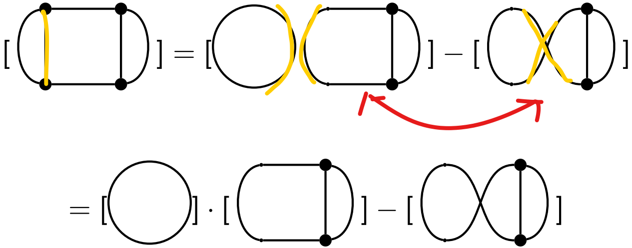
The number of proper 3-colorings of a planar cubic graph $[G]$ *follows the bracket relations:*

$$[O] = 3$$

$$[G_1 \sqcup G_2] = [G_1] \cdot [G_2]$$



Penrose Bracket (Worked Example)



$$= 3 \cdot 6 - 6 = 12$$

Failure of the Penrose evaluation for $K_{3,3}$

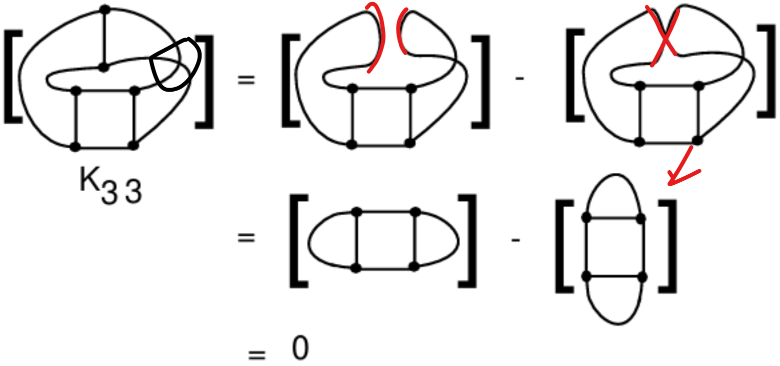
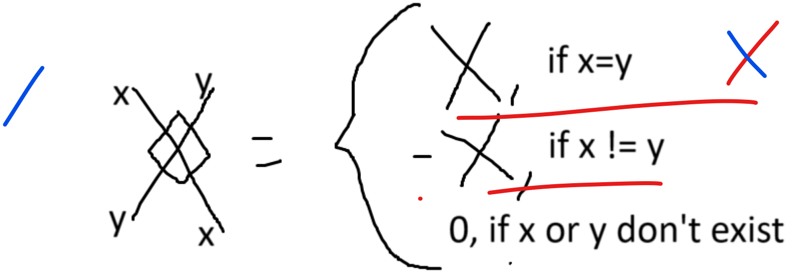


figure from (Kauffman 2025)

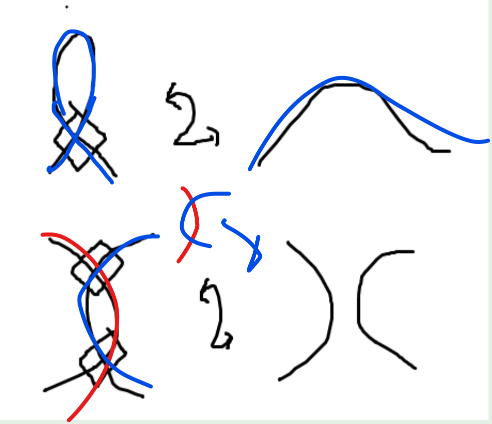
A new tensor

The failure of the Penrose evaluation is because the formations can cross each other at points not counted by epsilon tensors. Because the amount the formations cross eachother is still even, this can be fixed by introducing a new tensor at the initial crossings (interpreted as a different kind of virtual crossing):



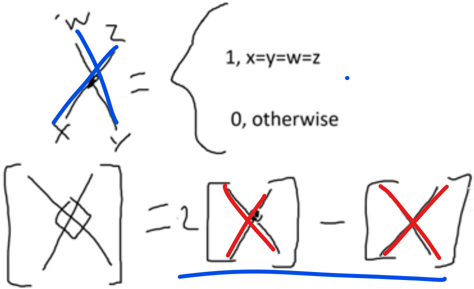
A new tensor

Example (Multivirtual Reidemeister moves I and II)



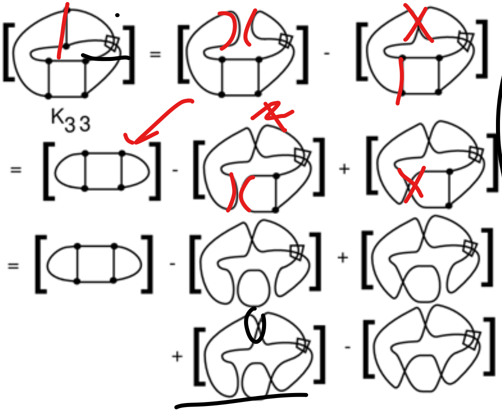
The Extended Penrose Bracket

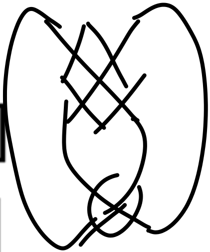
By introducing another tensor/crossing called a "node" or "fused crossing" a bracket relation for the multivirtual crossing can be obtained:



This is the motivation for the "chromatic bracket" introduced in [Kauf25].

Extended Penrose Bracket on $K_{3,3}$





$= 6 - 9$

$= -3$

$= 12 - 9 + 3$
$+3 - (3 - 6) = 12 - 9 + 3 + 3 - 3 + 6 = 12.$

figure from (Kauffman 2025)

References I

L.H. Kauffman
Multi-virtual knot theory.
Journal of Knot Theory and it's Ramifications, 2025.

L.H. Kauffman
Map coloring and the vector cross product
J Comb Theo B vol 48, no 2, April (1990) 145-154