

The Penrose-Kauffman Polynomial and Chromatic Bracket

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Outline

- 1 From the Penrose-Kauffman to the Chromatic Polynomial
- 2 More on the Chromatic Bracket
- 3 A application of the Chromatic Bracket

The Penrose-Kauffman Polynomial

Previously, bracket relations to count the number of proper 3-colorings of a cubic graph were defined (O should technically be a connected component without vertices instead of a loop):

$$[O] = 3 \quad (1)$$

$$[G_1 \sqcup G_2] = [G_1] \cdot [G_2] \quad (2)$$

$$\left[\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \end{array} \right] = \left[\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \quad \text{---} \end{array} \right] - \left[\begin{array}{c} \diagdown \quad \diagup \\ \text{---} \quad \text{---} \end{array} \right] \quad (3)$$

$$\left[\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right] = \left[\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \quad \text{---} \end{array} \right] - \left[\begin{array}{c} \diagdown \quad \diagup \\ \text{---} \quad \text{---} \end{array} \right] \quad (4)$$

A natural question to ask is what happens if we let the loop in (1) evaluate to any integer δ ?

The Penrose-Kauffman Polynomial

One interesting property is that the evaluations for $\delta = 3, -2$ (w.r.t. the perfect matching blow up) are proportional up to a function of the number of vertices (Penrose 1971) and (Baldrige, McCarty 2024). This polynomial is not well defined in general for a cubic graph.

$$\begin{aligned}
 \left[\text{diagram of a vertex with two loops} \right] &= [00] - [\infty] \\
 &= [0][0] - [0] = \delta^2 - \delta
 \end{aligned}$$

The Penrose-Kauffman Polynomial

$$\begin{aligned}
 \left[\begin{array}{c} \text{rectangle with horizontal crossings} \end{array} \right] &= \left[\begin{array}{c} \text{loop} \end{array} \right] \left[\begin{array}{c} \text{vertical line with crossings} \end{array} \right] - \left[\begin{array}{c} \text{figure-eight} \end{array} \right] \\
 &= \tau(\tau^2 - \tau) - \tau^2 + \tau = \tau(\tau - 1)^2
 \end{aligned}$$

$$\begin{aligned}
 \left[\begin{array}{c} \text{rectangle with colored crossings} \end{array} \right] &= \left[\begin{array}{c} \text{figure-eight with crossing} \end{array} \right] - \left[\begin{array}{c} \text{figure-eight with crossing} \end{array} \right] \\
 &= \left[\begin{array}{c} \text{figure-eight} \end{array} \right] - \left[\begin{array}{c} \text{figure-eight} \end{array} \right] + \left[\begin{array}{c} \text{figure-eight} \end{array} \right] \\
 &= \tau^2 - \tau - \tau + \tau^2 = \underline{2\tau(\tau - 1)}
 \end{aligned}$$

The Penrose-Kauffman Polynomial

Definition (Perfect Matching)

A perfect matching of a cubic graph is a set of disjoint edges that includes every vertex. Perfect matching edges are denoted by a slashed line

Definition (Penrose-Kauffman Polynomial)

The Penrose-Kauffman Polynomial $PK(G, M)$ with respect to graph G and a perfect matching M is the polynomial defined by (1),(2),(4) with relation (3) being applied to perfect matching edges.

It is possible to define a canonical perfect matching on a "blow-up" graph.

The Penrose-Kauffman Polynomial

Theorem

The Penrose-Kauffman Polynomial with respect to a perfect matching can be interpreted as the number of proper n -colorings, with each perfect matching edge being uncolored and attached to edges with only 2 colors.

A proof can be found in (Kauffman 2025) p. 27-28. The appendix of (Baldrige, Kauffman, McCarty 2025) includes a Mathematica program to calculate the Penrose-Kauffman Polynomial.

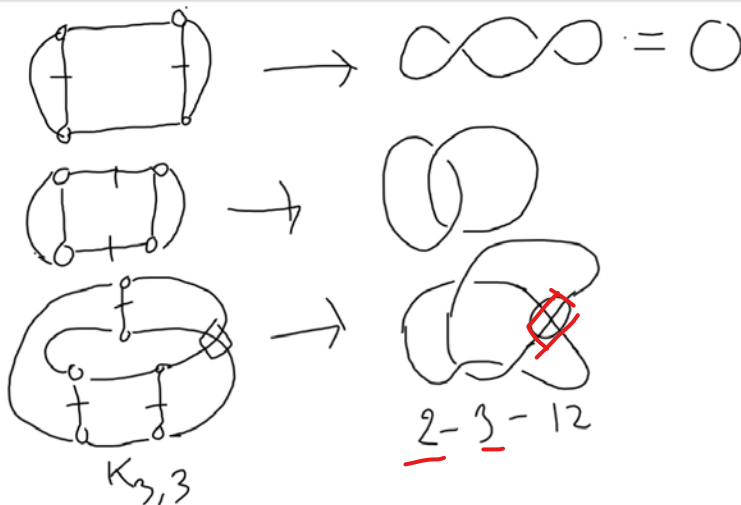
From Perfect Matching Graphs to Knots

A perfect matching graph can be associated with a knot by the following rule:



This does not give a correspondence between multivirtual knots and graphs. (Baldrige, Kauffman, Rushworth 2022) defines a strong correspondence between cubic perfect matching graph with extra information: "Graphenes" and virtual knots.

From Perfect Matching Graphs to Knots



From Perfect Matching Graphs to Knots

One can generalize the Bracket relations for the PK Polynomial with A, B variables much like the Kauffman bracket, although this does not yield the chromatic bracket...

$$P(G, M)(\text{Y-junction}) = AP(G, M)(\text{empty}) + BP(G, M)(\text{X-junction}),$$

$$P(G, M)(O) = \delta,$$

$$P(G, M)(\text{X-junction}) = 2P(G, M)(\text{Y-junction}) - P(G, M)(\text{empty}).$$

from (Kauffman 2025)

$$\langle \text{X-junction} \rangle = A \langle \text{empty} \rangle + B \langle \text{Y-junction} \rangle$$

$$\langle \text{Y-junction} \rangle = A \langle \text{empty} \rangle + B \langle \text{X-junction} \rangle$$

Invariance of the Chromatic Bracket & Fused Crossing moves

The Chromatic Bracket was introduced previously with the following relations:

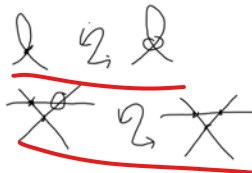
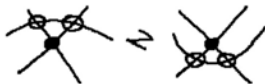
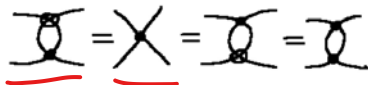
$$\langle \times \rangle = A \langle \smile \rangle + B \langle \rangle \langle \rangle$$

$$\langle O \rangle = \delta$$

$$\langle \otimes \rangle = 2 \langle \bigcirc \rangle - \langle \times \rangle$$

For a topological specialization, invariance under classical Reidemeister moves will result in the relations $B = A^{-1}$, $\delta = -A^2 - A^{-2}$ and normalization constant, but fused crossing moves will be needed for multi-virtual reidemeister moves.

Invariance of the Chromatic Bracket & Fused Crossing moves



Invariance of the Chromatic Bracket & Fused Crossing moves

$$\begin{aligned}
 & \text{Diagram 1} \rightarrow \underline{2 \text{ Diagram 2}} - \text{Diagram 3} = 2 \text{ Diagram 4} - \text{Diagram 5} = \text{Diagram 6} \\
 & \text{Diagram 7} \rightarrow 2 \text{ Diagram 8} - \text{Diagram 9} \\
 & \rightarrow 4 \text{ Diagram 10} - 2 \text{ Diagram 11} - 2 \text{ Diagram 12} + \text{Diagram 13} \\
 & = \text{Diagram 14} = \text{Diagram 15}
 \end{aligned}$$

Note: A red arrow points from the coefficient 4 in the third row to the coefficient 2 in the second row. A red '0' is written next to the final result.

from (Kauffman 2025)

Invariance of the Chromatic Bracket & Fused Crossing moves

The image shows a handwritten derivation of the invariance of the chromatic bracket under fused crossing moves. It consists of several lines of equations involving knot diagrams and their algebraic representations.

The first line shows a diagram of two crossings being equal to a linear combination of four diagrams, with a red '11' written next to the second term in the bracket.

The second line shows a similar expansion for a different crossing configuration, with a red '10' written next to the second term in the bracket.

Red arrows indicate the relationship between the two main equations, showing that they are equivalent after simplification.

The final line shows the result of the simplification, where the two original diagrams are shown to be equal.

from (Kauffman 2025)

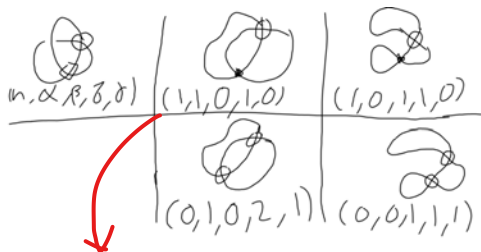
State Sum for the Chromatic Bracket

A sum states can be defined for the chromatic bracket much like the (unnormalized) Kauffman bracket (δ is now d):

$$\langle K \rangle[A, B, d] = \sum_s (-1)^\gamma 2^n \underline{A^{\alpha(s)} B^{\beta(s)} d^\delta}$$

Where n is the # of nodes, and γ is the difference between the original # of (ordinary) virtual crossings and (ordinary) virtual crossings in the state. Notice that multi-virtual knots that differ only by type of virtual crossing will have shared terms in the sum.

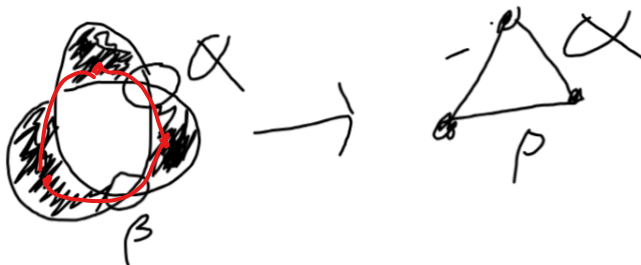
State Sum for the Chromatic Bracket



$$\text{So } \langle K \rangle[A, B, d] = 2Ad + 2Bd - Ad^2 - Bd$$

State Sum for the Chromatic Bracket

Since there are difficulties in generalizing the virtual thistlethwaite theorem through state sums, a possible alternative is to adapt the approach of (Diao, Heteyi 2010) using the tutte polynomial on colored graphs.

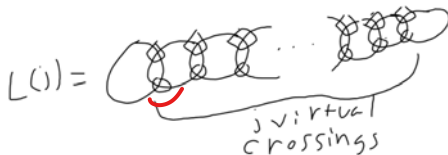


The tabulation of virtual knots vs. multivirtual knots

In virtual knot theory, the number of virtual knots/links for a given classical crossing number is bounded. Consequently, virtual knots/links are tabulated by classical crossing number. The following construction of (Kauffman, Mukherjee, Vojtěchovský 2025) shows that this is not the case for multivirtual knots/links:

The tabulation of virtual knots vs. multivirtual knots

The following infinite family of multivirtual links $L(j)$:



has no classical crossings and consequently no A or B smoothings. The highest degree term for the chromatic bracket polynomial will come from the state with most connected components $(j+1)$. Therefore the highest degree term of $\langle L(j) \rangle$ is A^{2j+2} and $L(j)$ constitutes a infinite family of multivirtual links with 0 classical crossings.

links

References I



S. Baldridge, L.H. Kauffman, W. Rushworth

On ribbon graphs and virtual links

[European J. Combin. 103 \(2022\)](#)



S. Baldridge, B. McCarty

A State Sum for the Total Face Color Polynomial

[Journal of Graph Theory, 109 \(2025\): 481-491](#)



S. Baldridge, B. McCarty

Quantum state systems that count perfect matchings

[arXiv:2303.12010](#)

References II



Y. Diao, G. Heteyi

Relative Tutte polynomials for colored graphs and virtual knot theory

[Comb. Probab. Comput. 19, 343–369 \(2010\)](#)



L.H. Kauffman

Multi-virtual knot theory.

[Journal of Knot Theory and it's Ramifications, 2025.](#)



L.H. Kauffman, S. Mukherjee, P. Vojtěchovský

Algebraic invariants of multi-virtual links

[arXiv:2504.09368](#)

References III



R. Penrose

Applications of negative dimensional tensors

Combinatorial Mathematics and Its Applications