The Penrose-Kauffman Polynomial and Chromatic Bracket

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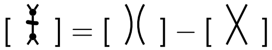
Outline

The Penrose-Kauffman Polynomial

Previously, bracket relations to count the number of proper 3-colorings of a cubic graph were defined (O should technically be a connected component without vertices instead of a loop):

$$[O] = 3 \tag{1}$$

$$[G_1 \sqcup G_2] = [G_1] \cdot [G_2] \tag{2}$$



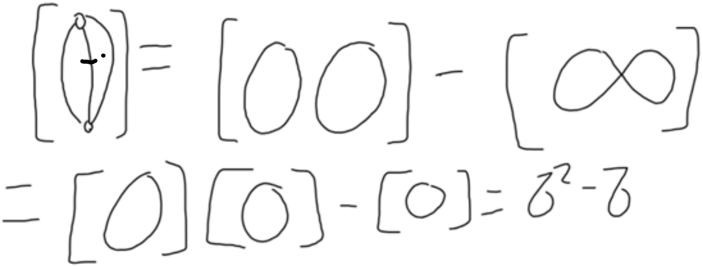
(3)

(4)

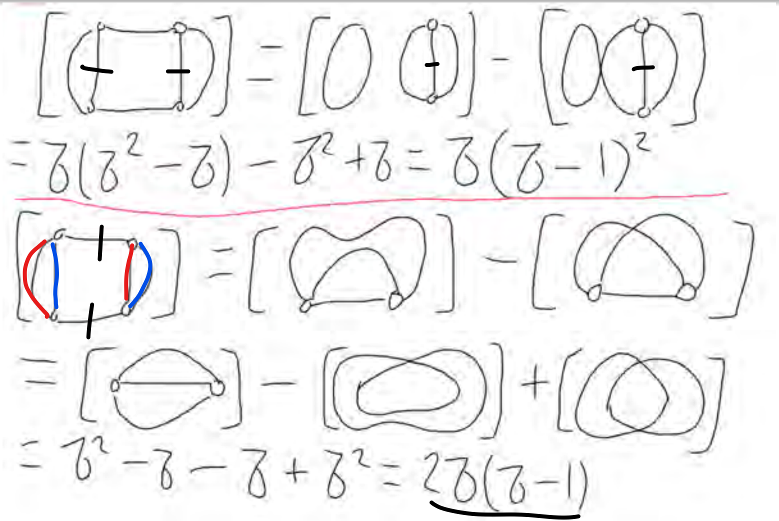
A natural question to ask is what happens if we let the loop in (1) evaluate to any integer δ?

The Penrose-Kauffman Polynomial

One interesting property is that the evaluations for $\delta = 3, -2$ (w.r.t. the perfect matching blow up) are proportional up to a function of the number of vertices (Penrose 1971) and (Baldridge, McCarty 2024). This polynomial is not well defined in general for a cubic graph.



The Penrose-Kauffman Polynomial



The Penrose-Kauffman Polynomial

Definition (Perfect Matching)

A perfect matching of a cubic graph is a set of disjoint edges that includes every vertex. Perfect matching edges are denoted by a slashed line

Definition (Penrose-Kauffman Polynomial)

The Penrose-Kauffman Polynomial $PK(G, M)$ with respect to graph G and a perfect matching M is the polynomial defined by (1),(2),(4) with relation (3) being applied to perfect matching edges.

It is possible to define a canonical perfect matching on a "blow-up" graph.

The Penrose-Kauffman Polynomial

Theorem

The Penrose-Kauffman Polynomial with respect to a perfect matching can be intepreted as the number of proper n-colorings, with each perfect matching edge being uncolored and attached to edges with only 2 colors.

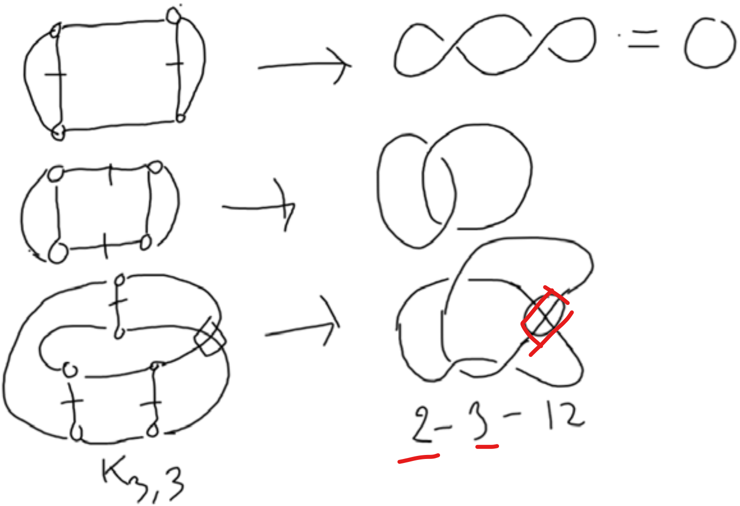
A proof can be found in (Kauffman 2025) p. 27-28. The appendix of (Baldridge, Kauffman, McCarty 2025) includes a Mathematica program to calculate the Penrose-Kauffman Polynomial.

From Perfect Matching Graphs to Knots

A perfect matching graph can be associated with a knot by the following rule:

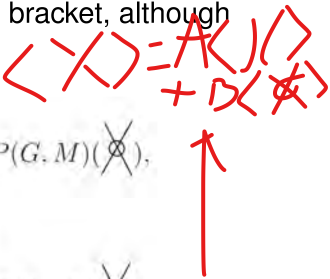
This does not give a correspondence between multivirtual knots and graphs. (Baldridge, Kauffman, Rushworth 2022) defines a strong correspondence between cubic perfect matching graph with extra information: "Graphenes" and virtual knots.

From Perfect Matching Graphs to Knots



From Perfect Matching Graphs to Knots

One can generalize the Bracket relations for the PK Polynomial with A, B variables much like the Kauffman bracket, although this does not yield the chromatic bracket...

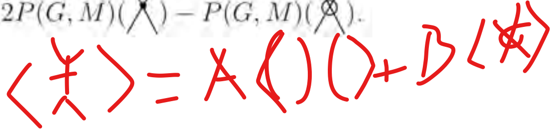


$$P(G,M)(\asymp) = AP(G,M)(\,)\,(\,) + BP(G,M)(\otimes),$$

$$P(G,M)(O) = \delta,$$

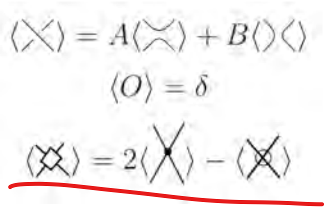
$$P(G,M)(\ast) = 2P(G,M)(\times) - P(G,M)(\otimes).$$

from (Kauffman 2025)



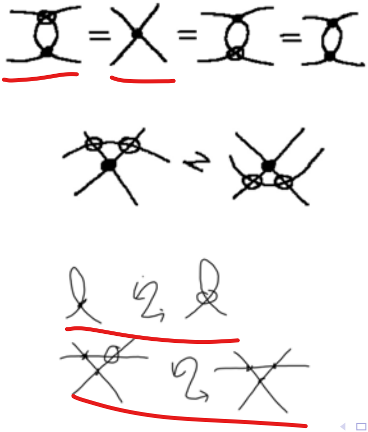
Invariance of the Chromatic Bracket & Fused Crossing moves

The Chromatic Bracket was introduced previously with the following relations:

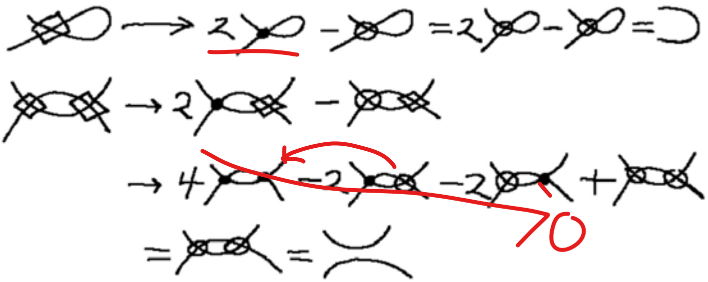


For a topoligcal specialization, invariance under classical Reidemeister moves will result in the relations $B = A^{-1}$, $\delta = -A^2 - A^{-2}$ and normalization constant, but fused crossing moves will be needed for multi-virtual reidemeister moves.

Invariance of the Chromatic Bracket & Fused Crossing moves

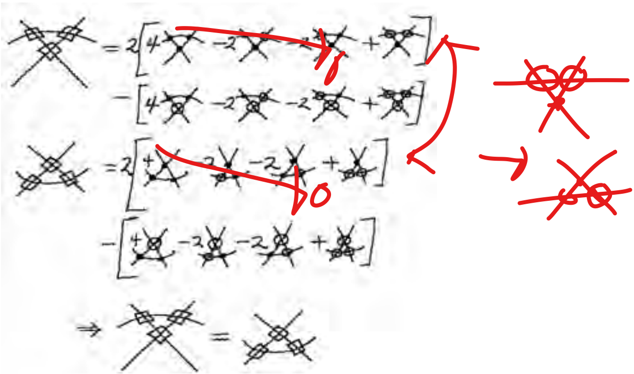


Invariance of the Chromatic Bracket & Fused Crossing moves



from (Kauffman 2025)

Invariance of the Chromatic Bracket & Fused Crossing moves



from (Kauffman 2025)

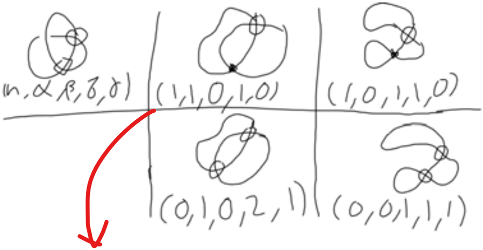
State Sum for the Chromatic Bracket

A sum states can be defined for the chromatic bracket much like the (unnormalized) Kauffman bracket (δ is now d):

$$\langle K\rangle[A,B,d]=\sum_{s}(-1)^{\gamma}2^{n}A^{\alpha(s)}B^{\beta(s)}d^{\delta}$$

Where n is the # of nodes, and γ is the difference between the original # of (ordinary) virtual crossings and (ordinary) virtual crossings in the state. Notice that multi-virtual knots that differ only by type of virtual crossing will have shared terms in the sum.

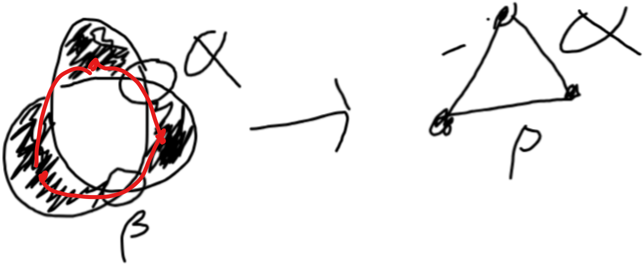
State Sum for the Chormatic Bracket



So $\langle K \rangle [A, B, d] = 2Ad + 2Bd - Ad^2 - Bd$

State Sum for the Chromatic Bracket

Since there are difficulties in generalizing the virtual thistlethwaite theorem through state sums, a possible alternative is to adapt the approach of (Diao, Hetyei 2010) using the tutte polynomial on colored graphs.

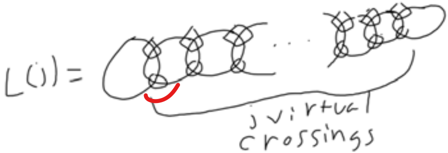


The tabulation of virtual knots vs. multivirtual knots

In virtual knot theory, the number of virtual knots/links for a given classical crossing number is bounded. Consequently, virtual knots/links are tabulated by classical crossing number. The following construction of (Kauffman, Mukherjee, Vojtěchovský 2025) shows that this is not the case for multivirtual knots/links:

The tabulation of virtual knots vs. multivirtual knots

The following infinite family of multivirtual links $L(j)$:



has no classical crossings and consequently no A or B smoothings. The highest degree term for the chromatic bracket polynomial will come from the state with most connected components (j+1). Therefore the highest degree term of $\langle L(j) \rangle$ is A^{2j+2} and $L(j)$ constitutes a infinite family of multivirtual ~~knot~~ links with 0 classical crossings.

References I

S. Baldridge, L.H. Kauffman, W. Rushworth
On ribbon graphs and virtual links
European J. Combin. 103 (2022)

S. Baldridge, B. McCarty
A State Sum for the Total Face Color Polynomial
Journal of Graph Theory, 109 (2025): 481-491

S. Baldridge, B. McCarty
Quantum state systems that count perfect matchings
arXiv:2303.12010

References II

Y. Diao, G. Hetyei
Relative Tutte polynomials for colored graphs and virtual knot theory
Comb. Probab. Comput. 19, 343–369 (2010)

L.H. Kauffman
Multi-virtual knot theory.
Journal of Knot Theory and it's Ramifications, 2025.

L.H. Kauffman, S. Mukherjee, P. Vojtěchovský
Algebraic invariants of multi-virtual links
arXiv:2504.09368

References III

R. Penrose
Applications of negative dimensional tensors
Combinatorial Mathematics and Its Applications